On homotopy invariants of combings of 3–manifolds

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Abstract

Combings of oriented compact 3-manifolds are homotopy classes of nowhere zero vector fields in these manifolds. A first known invariant of a combing is its Euler class, that is the Euler class of the normal bundle to a combing representative in the tangent bundle of the 3-manifold M. It only depends on the Spin^c-structure represented by the combing. When this Euler class is a torsion element of $H^2(M;\mathbb{Z})$, we say that the combing is a torsion combing. Gompf introduced a $\mathbb{Q}$-valued invariant θ_G of torsion combings of closed 3-manifolds that distinguishes all combings that represent a given Spin^c-structure. This invariant provides a grading of the Heegaard Floer homology $\widehat{HF}$ for manifolds equipped with torsion Spin^c-structures. We give an alternative definition of the Gompf invariant and we express its variation as a linking number. We also define a similar invariant p_1 for combings of manifolds bounded by S^2. We show that the Θ-invariant, that is the simplest configuration space integral invariant of rational homology spheres, is naturally an invariant of combings of rational homology balls, that reads $(\frac{1}{4}p_1 + 6\lambda)$ where λ is the Casson-Walker invariant. The article also includes a mostly self-contained presentation of combings.

Keywords: Spin^c-structure, nowhere zero vector fields, first Pontrjagin class, Euler class, homology 3–spheres, Heegaard Floer homology grading, Gompf invariant, Theta invariant, Casson-Walker invariant, perturbative expansion of Chern-Simons theory, configuration space integrals
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1 Introduction

1.1 General introduction

In this article, M is an oriented connected compact smooth 3-manifold. The boundary ∂M of M is either empty or identified with the unit sphere S^2 of $\mathbb{R}^3$. In this latter case, a neighborhood

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$N(\partial M)$ of ∂M in M is identified with a neighborhood of S^2 in the unit ball of $\mathbb{R}^3$. The tangent bundle of M is denoted by TM, and the unit tangent bundle of M is denoted by UM. Its fiber is $U_mM = (T_mM \setminus \{0\})/\mathbb{R}^{+*}$. All parallelizations of M are assumed to coincide with the parallelization induced by the standard parallelization τ_s of $\mathbb{R}^3$ over $N(\partial M)$, and all sections of UM are assumed to be constant with respect to this parallelization over $N(\partial M)$. Homotopies of parallelizations or sections satisfy these assumptions at any time. When $\partial M = \emptyset$, the parallelizations of M also induce the orientation of M.

A *combing* of M is a homotopy class of such sections of UM. According to Turaev [Tur97], a *Spin^c-structure* on M may be seen as an equivalence class of sections of UM, where two sections are in the same class if and only if they are homotopic over the complement of a point that sits in the interior of M.

For $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$, a $\mathbb{K}$-*sphere or (integral or rational) homology sphere* (resp. a $\mathbb{K}$-*ball*) is a smooth, compact, oriented 3-manifold with the same $\mathbb{K}$-homology as the sphere S^3 (resp. as a point).

In this mostly self-contained article, we study the combings of M, that are homotopy classes of sections of UM. We first describe the first known homotopy invariant of a combing, that is the Euler class, in terms of linking numbers. The *Euler class* of a combing is the Euler class of the normal bundle to a combing representative in TM. It only depends on the Spin^c-structure induced by the combing. When this Euler class is a torsion element of $H^2(M, \partial M; \mathbb{Z})$, we say that the combing is a *torsion combing*. We introduce a rational invariant p_1 of torsion combings of M. When M is *closed* (i. e. compact, without boundary), we show that the invariant p_1 coincides with the Gompf invariant θ_G. Thus, $\frac{\theta_G+2}{4}$ coincides with the Ozsváth Szabó grading of the Heegaard-Floer homology $\widehat{HF}$ for manifolds equipped with torsion Spin^c-structures in [OS06], according to a Gripp and Huang article [GH11]. For a combing that extends to a parallelization, the invariant p_1 coincides with the Pontrjagin number (or Hirzebruch defect) of the parallelization, studied in [KM99, Les04a, Les12]. In general, we express the variation of p_1 in terms of linking numbers. We also show that the Θ-invariant, that is the simplest configuration space integral invariant of rational homology spheres, is naturally another canonical invariant of combings of rational homology balls. It reads $(\frac{1}{4}p_1 + 6\lambda)$ where λ is the Casson-Walker invariant normalized like in [AM90, Mar88] for $\mathbb{Z}$-spheres and like $\frac{\lambda_W}{2}$ for $\mathbb{Q}$-spheres, where λ_W is the Walker normalisation in [Wal92].

1.2 Conventions and notations

Unless otherwise mentioned, all manifolds are oriented. Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients chains or manifolds. The fiber of the normal bundle $N(A)$ of an oriented submanifold A is oriented so that the normal bundle followed by the tangent bundle of the submanifold induce the orientation of the ambient manifold, fiberwise. The transverse intersection of two submanifolds A and B in a manifold M is oriented so that the normal bundle of $A \cap B$ is $(N(A) \oplus N(B))$, fiberwise. If the

two manifolds are of complementary dimensions, then the sign of an intersection point is $+1$ if the orientation of its normal bundle coincides with the orientation of the ambient space, that is if $T_xM = N_xA \oplus N_xB$ (as oriented vector spaces), this is equivalent to $T_xM = T_xA \oplus T_xB$. Otherwise, the sign is -1. If A and B are compact and if A and B are of complementary dimensions in M, their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by $\langle A, B\rangle_M$. The *linking number* of two rationally null-homologous disjoint knots in a 3-manifold is the algebraic intersection of a rational chain bounded by one of the knots and the other one. In this article, *blowing up* a submanifold A means replacing it by its unit normal bundle. Locally, $((\mathbb{R}^c = \{0\} \cup]0, \infty[\times S^{c-1}) \times A)$ is replaced by $([0, \infty[\times S^{c-1} \times A)$. Topologically, this amounts to removing an open tubular neighborhood of the submanifold (thought of as infinitely small), but the process is canonical, so that the created boundary is the unit normal bundle of the submanifold and there is a canonical projection from the blown-up manifold to the initial manifold.

1.3 Expanded introduction

Let us now be more explicit in order to state the main results precisely. The assertions below will be justified in Subsections 2.2 and 2.3. When a parallelization τ of M is given, two sections X and Y of UM induce a map $(X, Y)\colon M \to S^2 \times S^2$. Two sections X and Y are said to be *transverse* if the induced maps (X, Y) and $(X, -Y)$ are transverse to the diagonal of $S^2 \times S^2$, that is if their images are. This is generic and independent of τ. For two transverse sections X and Y, let $L_{X=Y}$ be the preimage of the diagonal of S^2 under the map (X, Y). Thus $L_{X=Y}$ is an oriented link in the interior of M. It is cooriented by the fiber of the normal bundle to the diagonal of $(S^2)^2$.

The Spin^c-structures of M form an affine space $\mathcal{S}(M)$ with translation group $H^2(M, \partial M; \pi_2(S^2))$. The Poincaré duality isomorphism $P\colon H^2(M, \partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ identifies the translation group of $\mathcal{S}(M)$ with $H_1(M; \mathbb{Z})$. Then, for any two transverse sections X and Y of UM, the difference $([X]^c - [Y]^c) \in H_1(M; \mathbb{Z})$ of the two corresponding Spin^c-structures $[X]^c$ and $[Y]^c$ of M is the class of $L_{X=-Y}$ in $H_1(M; \mathbb{Z})$.

The Euler class of a combing $[X]$ represented by a section X, is the Euler class of the normal bundle of $TM/\mathbb{R}X$. It is denoted as $e(X^\perp)$, it belongs to $H^2(M; \mathbb{Z})$ (here, $H^2(M; \mathbb{Z}) = H^2(M, \partial M; \mathbb{Z})$) and satisfies

$$P(e(X^\perp)) = [X]^c - [-X]^c$$

so that for two combings $[X]$ and $[Y]$,

$$P(e(X^\perp) - e(Y^\perp)) = 2([X]^c - [Y]^c).$$

A *torsion combing* of M is a combing whose Euler class is a torsion element of $H^2(M, \partial M; \mathbb{Z})$. A *torsion section* of UM is a section that represents a torsion combing.

There is a natural transitive action of $\pi_3(S^2) = \mathbb{Z}$ on the combings of M that belong to a given Spin^c-structure. This action is free for *torsion Spin^c-structures*, that are Spin^c-structures represented by torsion combings.

We shall prove the following theorem in Subsection 3.2.

Theorem 1.1 *Let X be a fixed section of UM. Two sections Y and Y' of UM transverse to X represent the same Spin^c-structure if and only if the links $L_{Y=-X}$ and $L_{Y'=-X}$ are homologous.*

If X is a torsion section, then a section Y of UM transverse to X is a torsion section if and only if the links $L_{Y=-X}$ and $L_{Y=X}$ are rationally null-homologous in M.

If X is a torsion section, then two torsion sections Y and Y' of UM transverse to X represent the same combing if and only if the links $L_{Y=-X}$ and $L_{Y'=-X}$ are homologous, and $lk(L_{Y=-X}, L_{Y=X}) = lk(L_{Y'=-X}, L_{Y'=X})$.

This theorem is a generalisation of a Pontrjagin theorem recalled in Subsection 2.2 that treats the case when X extends to a trivialization.

The first Pontrjagin class induces a canonical map p_1 from the set of parallelization homotopy classes of M to $\mathbb{Z}$, that is studied under the name *Hirzebruch defect* in [KM99] when $\partial M = \emptyset$, by Kirby and Melvin, and studied and used as p_1 in [Les04a, Les12] when M is a $\mathbb{Q}$-ball. The definition of p_1 and some of its properties are recalled in Subsection 2.1.

The main result of this article is the following theorem that is proved in Section 3.

Theorem 1.2 *There exists a unique map*

$$p_1 \colon \{\textit{Torsion combings of } M\} \to \mathbb{Q}$$

such that

- *if the combing $[X]$ extends as a parallelization τ, then $p_1([X]) = p_1(\tau)$,*
- *for any two transverse torsion sections X and Y of UM,*

$$p_1([Y]) - p_1([X]) = 4lk(L_{X=Y}, L_{X=-Y}),$$

The map p_1 satisfies the following properties:

- *For any combing $[X]$, $p_1([X]) = p_1([-X])$.*
- *The restriction of p_1 to any torsion Spin^c-structure is injective.*

We present simple operations on combings in Definition 2.6. The variation of p_1 under these and other simple operations on torsion combings are presented in Subsection 4.1, and the image of p_1 is determined by the following theorem that is also proved in Subsection 4.1.

Let $\ell \colon \mathrm{Torsion}(H_1(M;\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}$ denote the *self-linking number* (the linking number of a representative and one of its parallels). View an element $\overline{a}$ of $\mathbb{Q}/\mathbb{Z}$ as its class $(a + \mathbb{Z})$ in $\mathbb{Q}$ so that $4\ell(\mathrm{Torsion}(H_1(M;\mathbb{Z}))$ is a subset of $\mathbb{Q}$, invariant by translation by 4.

Theorem 1.3 *Let τ be a parallelization of M inducing a combing X. For any torsion combing Y,*

$$p_1(Y) \in (p_1(\tau) - 4\ell([L_{Y=-X}])).$$

$$p_1(\{\text{Torsion combings}\}) = p_1(\tau) - 4\ell(\text{Torsion}(H_1(M;\mathbb{Z})).$$

Here $p_1(\tau)$ is an integer whose parity is determined in Theorem 2.3 below. Note that the image of p_1 is not an affine space in general.

In Subsection 4.2, we prove that the invariant p_1 coincides with the Gompf invariant when $\partial M = \emptyset$. The Gompf invariant is denoted by θ in [Gom98], and it is denoted by θ_G in this article to prevent confusion with Θ.

In [OS04, Section 2.6], Ozsváth and Szabó associate a Spin^c-structure to a generator $\mathbf{x}$ of the Heegaard Floer homology $\widehat{HF}$. Gripp and Huang refine this process in [GH11] in order to associate a combing $\tilde{gr}(\mathbf{x})$ to such a generator $\mathbf{x}$, and they relate the Gompf invariant with the absolute $\mathbb{Q}$-grading $\overline{gr}$ of Ozsváth and Szabó for the Heegaard Floer homology of 3-manifolds equipped with torsion Spin^c structures in [OS06]. According to [GH11, Corollary 4.3], $\overline{gr}(\mathbf{x}) = \frac{2+\theta_G(\tilde{gr}(\mathbf{x}))}{4}$.

The work of Witten [Wit89] pioneered the introduction of many $\mathbb{Q}$-sphere invariants, among which the Kontsevich configuration space invariant [Kon94] that was proved to be equivalent to the LMO invariant of Le, Murakami and Ohtsuki [LMO98] for integral homology spheres by G. Kuperberg and D. Thurston [KT99]. This Kontsevich configuration space invariant is in fact an invariant of parallelised $\mathbb{Q}$-balls M. Its degree one part is called the Θ-invariant. For a $\mathbb{Q}$-ball M equipped with a parallelization τ, the invariant $\Theta(M,\tau)$ is the sum of $6\lambda(M)$ and $\frac{p_1(\tau)}{4}$, where λ is the Casson-Walker invariant, according to a Kuperberg-Thurston theorem [KT99] generalized to rational homology spheres in [Les04b, Theorem 2.6 and Section 6.5].

The extension of the map p_1 to torsion combings in Theorem 1.2 and the proof of the variation formula $p_1([Y]) - p_1([X]) = 4lk(L_{X=Y}, L_{X=-Y})$ occured to me when I realized that the Θ-invariant is actually an invariant of combings X of $\mathbb{Q}$-balls M such that

$$\Theta(M,X) = 6\lambda(M) + \frac{1}{4}p_1(X).$$

The Θ-invariant is presented as an invariant of combings in Section 5, and the above formula is proved in this section.

2 Preliminaries and background

2.1 The original map p_1 for parallelizations

It has long been known that smooth compact oriented 3-manifolds are parallelisable.

Let M be equipped with a parallelization $\tau_M\colon M\times\mathbb{R}^3 \to TM$. Let $GL^+(\mathbb{R}^3)$ denote the group of orientation-preserving linear isomorphisms of $\mathbb{R}^3$. Let $[(M,\partial M),(GL_+(\mathbb{R}^3),1)]_m$ denote the

set of maps

$$g : (M, \partial M) \longrightarrow (GL^+(\mathbb{R}^3), 1)$$

from M to $GL^+(\mathbb{R}^3)$ that send ∂M to the unit 1 of $GL^+(\mathbb{R}^3)$. Let $[(M, \partial M), (GL_+(\mathbb{R}^3), 1)]$ denote the group of homotopy classes of such maps, with the group structure induced by the multiplication of maps using the multiplication in $GL_+(\mathbb{R}^3)$. For a map g in $[(M, \partial M), (GL_+(\mathbb{R}^3), 1)]_m$, define

$$\begin{array}{llll}\psi(g) : & M \times \mathbb{R}^3 & \longrightarrow & M \times \mathbb{R}^3 \\ & (x, y) & \mapsto & (x, g(x)(y)).\end{array}$$

Then any parallelization τ of M that coincides with τ_M on ∂M reads

$$\tau = \tau_M \circ \psi(g)$$

for some $g \in [(M, \partial M), (GL_+(\mathbb{R}^3), 1)]_m$. Thus, fixing τ_M identifies the set of homotopy classes of parallelizations of M fixed on ∂M to the group $[(M, \partial M), (GL_+(\mathbb{R}^3), 1)]$. Since $GL_+(\mathbb{R}^3)$ deformation retracts onto the group $SO(3)$ of orientation-preserving linear isometries of $\mathbb{R}^3$, the group $[(M, \partial M), (GL_+(\mathbb{R}^3), 1)]$ is isomorphic to $[(M, \partial M), (SO(3), 1)]$.

See B^3 as the quotient of $[0, 2\pi] \times S^2$ where the quotient map identifies $\{0\} \times S^2$ to a point. Then the map from B^3 to $SO(3)$ that maps $(\theta \in [0, 2\pi], x \in S^2)$ to the rotation $\rho(\theta, x)$ with axis directed by x and with angle θ is denoted by ρ. It induces the standard double covering map $\tilde{\rho}$ from $S^3 = B^3/\partial B^3$ to $SO(3)$ that orients $SO(3)$. The following standard proposition is proved in [Les12].

Proposition 2.1 *For any compact connected oriented* 3*-manifold* M*, the group* $[(M, \partial M), (SO(3), 1)]$ *is abelian, and the degree*

$$deg\colon [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Z}$$

is a group homomorphism, that induces an isomorphism

$$deg\colon [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Let W be a compact 4–dimensional manifold with signature 0 whose boundary is $\partial W = M \cup_{1 \times \partial M} (-[0, 1] \times S^2) \cup_{0 \times S^2} (-B^3)$ (resp. $\partial W = M$) when $\partial M = S^2$ (resp. when $\partial M = \emptyset$). When $\partial M = S^2$, W has ridges and it is identified with an open subspace of one of the products $[0, 1[\times B^3$ or $]0, 1] \times M$ near ∂W. Then the *Pontrjagin number* $p_1(\tau)$ of a parallelization τ is the obstruction to extending the trivialization of $TW \otimes \mathbb{C}$ induced by τ and the standard parallelization τ_s of $\mathbb{R}^3$ on ∂W across W. This obstruction lives in $H^4(W, \partial W, \pi_3(SU(4)) = \mathbb{Z}) = \mathbb{Z}$. For more details, see [Les04a, Section 1.5] or [Les12] where the following classical theorem is proved.

Theorem 2.2 *Let* M *be a compact connected oriented* 3*-manifold such that* $\partial M = \emptyset$ *or* S^2*. For any map* g *in* $[(M, \partial M), (SO(3), 1)]_m$*, for any trivialization* τ *of* TM

$$p_1(\tau \circ \psi(g)) - p_1(\tau) = 2\,deg(g).$$

For $n \geq 3$, a *spin structure* of a smooth n–manifold is a homotopy class of parallelizations over a 2-skeleton of M (that is over the complement of a point when $n = 3$).

The class of the double covering map $\tilde{\rho}$ described above is the standard generator of $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$. The map ρ can be used to describe the action of $\pi_3(SO(3))$ on the homotopy classes of parallelizations ($\tau \colon M \times \mathbb{R}^3 \to TM$) of TM as follows. Let B be a 3–ball in M identified with B^3. Let $\tau\psi(\rho)$ coincide with τ outside $B \times \mathbb{R}^3$ and read $\tau \circ \psi(\rho)$ on $B \times \mathbb{R}^3$. Set $[\tilde{\rho}][\tau] = [\tau\psi(\rho)]$. The set of parallelizations that induce a given spin structure form an affine space with translation group $\pi_3(SO(3))$. According to Theorem 2.2, $p_1([\tilde{\rho}][\tau]) = p_1(\tau) + 4$.

The *Rohlin invariant* $\mu(M, \sigma)$ of a smooth closed 3-manifold M, equipped with a spin structure σ, is the mod 16 signature of a compact spin 4-manifold W bounded by M so that the spin structure of W restricts to M as a stabilisation of σ. The *first Betti number* of M that is the dimension of $H_1(M; \mathbb{Q})$ is denoted by $\beta_1(M)$.

Kirby and Melvin proved the following theorem [KM99, Theorem 2.6].

Theorem 2.3 *For any closed oriented 3-manifold M, for any parallelization τ of M,*

$$(p_1(\tau) - \mathit{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) - \beta_1(M)) \in 2\mathbb{Z}.$$

Let M be a closed 3-manifold equipped with a given spin structure σ, p_1 is a bijection from the set of homotopy classes of parallelizations of M that induce σ to

$$2\,(\mathit{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) + 1) + \mu(M, \sigma) + 4\mathbb{Z}$$

When M is a $\mathbb{Z}$–sphere, p_1 is a bijection from the set of homotopy classes of parallelizations of M to $(2 + 4\mathbb{Z})$.

Extend the standard parallelization τ_s of B^3 as a parallelization $\widehat{\tau_s}$ of S^3. When $\partial M = S^2$, we can form $\hat{M} = (S^3 \setminus (B^3 \setminus N(\partial M))) \cup_{N(\partial M)} M$ and use $\widehat{\tau_s}$ to extend any parallelization τ of M to a parallelization $\hat{\tau}$ of $\hat{M}$. Then it is easy to see that $p_1(\tau) = p_1(\hat{\tau}) - p_1(\widehat{\tau_s})$. In particular, according to Theorem 2.3, $(p_1(\tau) - \mathrm{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) - \beta_1(M)) \in 2\mathbb{Z}$ and, when M is a $\mathbb{Z}$–ball, the map p_1 is a bijection from the set of homotopy classes of parallelizations of M to $4\mathbb{Z}$.

2.2 Generalization of a Pontrjagin construction in dimension 3

Lemma 2.4 *Combings are generically transverse. For two transverse sections X and Y of UM, the homology classes of $L_{X=Y}$ and $L_{X=-Y}$ only depend on the Spin^c-structures $[X]^c$ and $[Y]^c$ represented by X and Y.*

PROOF: When X extends as a parallelization, this parallelization identifies UM with $M \times S^2$, then Y may be seen as a map from M to S^2, and a homotopy of Y is a map from $[0,1] \times M$ to S^2, for which X is a regular value, generically. In particular, the preimage of X under such

a homotopy h yields a cobordism from $L_{Y_0=X}$ and $L_{Y_1=X}$, and the homology class of $L_{Y=X}$ only depends on the homotopy class of Y, when X is fixed. Since any X locally extends as a parallelization, the local transversality arguments hold for any X so that the above proof may be adapted to any X by using a homotopy (Y_t, X) valued in $S^2 \times S^2$ (with respect to some reference trivialization) and the preimage of the diagonal under this homotopy. Similarly, the homology class of $L_{Y=-X}$ only depends on the homotopy classes of X and Y. Since the homology classes of $L_{Y=-X}$ and $L_{Y=X}$ are unchanged under a homotopy of X or Y supported in a ball, they only depend on $[X]^c$ and $[Y]^c$. ◇

Let X be a section of UM. Equip M with a Riemannian structure (all of these are homotopic). These two assumptions hold for the rest of the subsection.

Let NL be the normal bundle of a link L in M. Let $S(NL, (-X)^{\perp})$ denote the space of homotopy classes of sections of the S^1–bundle over L whose fiber over x is the space of orientation-preserving linear isometries from the fiber $N_xL = T_xM/T_xL$ of NL to $(-X(x))^{\perp} = T_xM/\mathbb{R}(-X(x))$.

An X*–framing* of L will be an element of $S(NL, (-X)^{\perp})$.

A *framing* of a link L of M is a homotopy class of sections of the unit normal bundle of L. Pushing L in the direction of such a section yields a *parallel* $L_{\parallel}$ of L up to isotopy of $L_{\parallel}$ in $N(L) \setminus L$, where $N(L)$ is a tubular neighborhood of L. This isotopy class of parallels induced by the framing also induces the framing. Thus, a *framing* of L is such an isotopy class of parallels of L.

Any section Y of UM transverse to X is equipped with an X–framing

$$\sigma(Y, X) \in S(NL_{Y=-X}, (-X)^{\perp})$$

of $L_{Y=-X}$, that is naturally induced by the restriction of the tangent map of Y.

Lemma 2.5 *Let X and Y be two transverse sections of UM. Let $N(L_{Y=-X})$ be a tubular neighborhood of $L_{Y=-X}$. There exists a section $\tilde{Y}$ homotopic to Y such that $L_{\tilde{Y}=-X} = L_{Y=-X}$, $\sigma(Y, X) = \sigma(\tilde{Y}, X)$ and $\tilde{Y}$ sends the complement of a tubular neighborhood of $N(L_{Y=-X})$ to X. Furthermore, the homotopy class of Y is determined by X, $L_{Y=-X}$ and $\sigma(Y, X)$.*

Proof: Outside $L_{Y=-X}$, there is a homotopy from Y to X. When $Y(m) \neq -X(m)$, there is a unique geodesic arc $[Y(m), X(m)]$ with length ($\ell \in [0, \pi[$) from $Y(m)$ to $X(m)$. For $t \in [0, 1]$, let $Y_t(m) \in [Y(m), X(m)]$ be such that the length of $[Y(m) = Y_0(m), Y_t(m)]$ is $t\ell$. Let D^2 be the unit disk of $\mathbb{C}$. Write $N(L_{Y=-X})$ as $D^2 \times L_{Y=-X}$, and let χ be a smooth increasing bijective function from $[0, 1]$ to $[0, 1]$ whose derivatives vanish at 0 and 1. Set $\tilde{Y}_t(m) = Y_t(m)$ if $m \notin N(L_{Y=-X})$ and $\tilde{Y}_t(v \in D^2, \ell \in L_{Y=-X}) = \begin{cases} Y_{\chi(|v|)t}(v, \ell) & \text{if } v \neq 0 \\ Y(0, \ell) & \text{if } v = 0. \end{cases}$
Then $\tilde{Y} = \tilde{Y}_1$ has the wanted properties.

Trivializing $(-X)^{\perp}$ over $N(L_{Y=-X})$, and reducing the size of $N(L_{Y=-X})$ allows one to prove that the homotopy class of Y is determined by X, $L_{Y=-X}$ and $\sigma(Y, X)$. ◇

We now present some combing modifications.

Definition 2.6 Let X be a section of UM. Let L be a link in the interior of M and let Z be a section of $UM_{|L}$ orthogonal to X. Let $\eta = \pm 1$, let $L_{\parallel}$ be a parallel of L and let $N(L)$ be a tubular neighborhood of L where Z is extended as a section of UM orthogonal to X. Let $\rho(\theta, X)$ denote the rotation with axis X and angle θ. Let $D^2 = \{u\exp(i\theta); u \in [0,1], \theta \in [0, 2\pi]\}$ be the unit disk of $\mathbb{C}$. Define $D(X, L, L_{\parallel}, Z, \eta)$ (up to homotopy) as the section of UM that coincides with X outside $N(L)$ and that reads as follows in $N(L)$ that is trivialized with respect to $L_{\parallel}$ so that it reads $D^2 \times L$.

- $D(X, L, L_{\parallel}, Z, \eta)(0, k \in L) = -X(0, k)$,
- when $u \in]0, 1]$, $[-X(u\exp(i\theta), k), D(X, L, L_{\parallel}, Z)(u\exp(i\theta), k)]$ is the geodesic arc of length $u\pi$ of the half great circle $[-X, X]_{\rho(\eta\theta, X)(Z)}$ from $(-X)$ to X through $\rho(\eta\theta, X)(Z)$, where X and Z stand for $X(u\exp(i\theta), k)$ and $Z(u\exp(i\theta), k)$, respectively,

so that $D(X, L, L_{\parallel}, Z, \eta)(1/2, k) = Z(1/2, k)$. Note that the homotopy class of $D(X, L, L_{\parallel}, Z, \eta)$ can also be defined by the following formula.

$$D(X, L, L_{\parallel}, Z, \eta)(u\exp(i\theta), k) = \rho(\pi(1+u), \rho(\eta\theta - \pi/2, X)(Z))(X)(u\exp(i\theta), k).$$

Let L be a link X–framed by some $[\sigma] \in S(NL, (-X)^{\perp})$ represented by $\sigma \colon N(L) \to (-X)^{\perp}$. Let σ_N be a unit section of $N(L)$ that induces a parallel $L_{\parallel}$ of L, up to isotopy. Set $Z(\sigma, \sigma_N)(x) = \sigma(x)(\sigma_N(x))$. Then $Z(\sigma, \sigma_N)$ is a section of $(-X)^{\perp}$.

Define
$$C(X, L, \sigma) = D(X, L, L_{\parallel}, Z(\sigma, \sigma_N), -1).$$

Remark 2.7 Note that $[\sigma]$ is determined by the homotopy classes of σ_N and $Z(\sigma, \sigma_N)$, where the homotopy class of σ_N may be replaced by the isotopy class of $L_{\parallel}$. Thus, we can think of elements of $S(NL, (-X)^{\perp})$ as pairs $(L_{\parallel}, Z(\sigma, \sigma_N))$ up to simultaneous twists of $L_{\parallel}$ and $Z(\sigma, \sigma_N)$.

Lemma 2.8 *Let X be a section of UM. Let L be a link equipped with an X–framing σ. Then $L_{C(X,L,\sigma)=-X} = L$ and $\sigma(C(X, L, \sigma), X) = \sigma$. For any section Y of UM transverse to X, Y is homotopic to $C(X, L_{Y=-X}, \sigma(Y, X))$. The Spin^c-structure of Y is determined by $[X]^c$ and $L_{Y=-X}$.*

Proof: The first properties are direct corollaries of Lemma 2.5. Let us prove that $[Y]^c = [C(X, L_{Y=-X}, \sigma(Y, X))]^c$ does not depend on the X-framing $\sigma(Y, X)$ of $L_{Y=-X}$. Two representatives σ_1 and σ_2 of any two X–framings of a link may be assumed to coincide over the link except over one little interval for each link component. Thus, the associated $C(X, L_{Y=-X}, \sigma_1)$ and $C(X, L_{Y=-X}, \sigma_2)$ coincide outside a finite union of balls that embeds in a larger ball. Then $[Y]^c$ is determined by X and $L_{Y=-X}$. Now, changing X inside its homotopy class or changing X over a ball does not affect $[Y]^c$. ◇

Let $(-X)^{\perp}$ also denote the pull-back of $(-X)^{\perp}$ under the natural projection from $[0, 1] \times M$ to M. Let Σ be a properly embedded surface in $[0, 1] \times M$. Let $S(N\Sigma, (-X)^{\perp})$ denote the

space of homotopy classes of sections of the S^1–bundle over Σ whose fiber over x is the space of orientation-preserving linear isometries from the fiber $N_x\Sigma = T_x([0,1]\times M)/T_x\Sigma$ of $N\Sigma$ to $(-X(x))^\perp$. An X*–framing* of Σ is an element of $S(N\Sigma,(-X)^\perp)$.

Two X–framed links L and L' are X*–framed cobordant* if and only if there exists an X–framed cobordism Σ (that is a cobordism equipped with an X-framing) properly embedded in $[0,1]\times M$, from $\{0\}\times L$ to $\{1\}\times L'$ that induces the X–framings of L and L'.

Theorem 2.9 *Let X be a section of UM. Two sections Y and Z of UM transverse to X are homotopic if and only if $(L_{Y=-X},\sigma(Y,X))$ and $(L_{Z=-X},\sigma(Z,X))$ are X-framed cobordant.*

Proof: View a homotopy Y_t from $Y=Y_0$ to $Z=Y_1$ as a section Y_t of the pull-back of UM under the natural projection from $[0,1]\times M$ to M, and assume without loss that $(Y_t,-X)$ is transverse to the diagonal of $S^2\times S^2$ (with respect to some trivialization). Then the preimage Σ of the diagonal is a cobordism from $L_{Y=-X}$ and $L_{Z=-X}$ that is canonically X–framed by an X-framing that induces those of $L_{Y=-X}$ and $L_{Z=-X}$.

Conversely, assume that there exists an X–framed cobordism Σ from $(L_{Y=-X},\sigma(Y,X))$ to $(L_{Z=-X},\sigma(Z,X))$. Write the tubular neghborhood of Σ as $D^2\times\Sigma$. With respect to this trivialization that induces a parallel $\Sigma_\parallel$ of Σ, the X-framing of Σ becomes a section Z of $(-X)^\perp$ over Σ. Define a unit section $D_\Sigma = D(X,\Sigma,\Sigma_\parallel,Z,-1)$ of the pull-back of TM under the natural projection from $[0,1]\times M$ to M like in Definition 2.6, so that D_Σ coincides with X outside $D^2\times\Sigma$ and

$$D_\Sigma(u\exp(i\theta),k\in\Sigma) = \rho(\pi(1+u),\rho(\eta\theta-\pi/2,X)(Z))(X)(u\exp(i\theta),k)$$

on $D^2\times\Sigma$. Then the restriction D_t of D_Σ on $\{t\}\times M$ defines a homotopy from $D_0 = C(X,L_{Y=-X},\sigma(Y,X))$ to $D_1 = C(X,L_{Z=-X},\sigma(Z,X))$, and, according to Lemma 2.8, Y and Z are homotopic. $\diamond$

Corollary 2.10 *Let X be a section of UM. The Spin^c–structure of a section Y of UM transverse to X is determined by $[X]^c$ and by the homology class $[L_{Y=-X}]$ of $L_{Y=-X}$ in $H_1(M;\mathbb{Z})$.*

$\diamond$

When X is the first vector of a parallelization τ, the second vector X_2 of τ is a section of $(-X)^\perp$, and τ identifies an X–framing $[\sigma]\in S(NL,(-X)^\perp)$ represented by σ with the isotopy class of parallels $L_\parallel$ of L induced by the section $\sigma^{-1}(X_2)$. Set

$$C(\tau,L,L_\parallel) = C(X,L,[\sigma])$$

and note that a parallelization τ with X as first vector identifies X-framings of links with framings of links. A *framed cobordism* from $(L,L_\parallel)$ to $(L',L'_\parallel)$ is a cobordism Σ from $\{0\}\times L$ to $\{1\}\times L'$ equipped a unit normal section of $T\Sigma$ in $T([0,1]\times M)$, up to homotopy, that induces

the given framings of L and L'. Two framed links are *framed cobordant* if and only if their exists a framed cobordism from one to the other one.

As above, a parallelization τ with X as first vector identifies X-framings of cobordisms to framings of cobordisms.

This allows us to state the following Pontrjagin theorem [Mil97, Section 7, Theorem B] as a corollary of Theorem 2.9.

Theorem 2.11 (Pontrjagin construction) *Let τ be a parallelization of M. Any section of UM is homotopic to $C(\tau, L, L_{\parallel})$ for a framed link $(L, L_{\parallel})$ of the interior of M. Two sections $C(\tau, L, L_{\parallel})$ and $C(\tau, L', L'_{\parallel})$ are homotopic if and only if $(L, L_{\parallel})$ and $(L', L'_{\parallel})$ are framed cobordant.*

$\diamond$

Pontrjagin proved generalisations of this theorem to every dimension. See [Mil97, Section 7].

Let Σ_M be an embedded cobordism from a link L to a link L_1 in M. The graph of a Morse function f from Σ_M to $[0,1]$ such that $f^{-1}(0) = L$ and $f^{-1}(1) = L_1$ yields a proper embedding Σ of Σ_M into $[0,1] \times M$. The positive normal of Σ_M in M at m seen in $T_{(f(m),m)}\{f(m)\} \times M$ frames Σ. This framing of Σ identifies the X-framings Σ with homotopy classes of sections of $(-X)^{\perp}$ over Σ. When Σ_M is connected, and when K is a boundary component of Σ, any X-framing defined on $\partial\Sigma \setminus K$ extends as an X-framing of Σ, and the extension of the X-framing over K is determined by the restriction of the X-framing to $\partial\Sigma \setminus K$.

Embed a sphere S with three holes in M, the 3 boundary components of S are 3 knots K_1, K_2 and $-K_1 \sharp_b K_2$ of M that are framed by the embedding of S. Then $K_1 \sharp_b K_2$ is a *framed band sum* of K_1 and K_2, it is framed cobordant to the union of K_1 and K_2. Note that any X–framed link is X–framed cobordant to an X–framed knot by such band sums. Similarly, any framed link is framed cobordant to a framed knot.

Lemma 2.12 *Two framed links $(L, L_{\parallel})$ and $(L', L'_{\parallel})$ in a $\mathbb{Z}$-sphere or in a $\mathbb{Z}$-ball are framed cobordant if and only if $lk(L, L_{\parallel}) = lk(L', L'_{\parallel})$.*

PROOF: When the framed links are framed cobordant, $lk(L, L_{\parallel}) = lk(L', L'_{\parallel})$, since $lk(L, L_{\parallel})$ is the algebraic intersection of two 2-chains bounded by $L \times \{0\}$ and $L_{\parallel} \times \{0\}$ in $[-1, 0] \times M$. Conversely, let $(L, L_{\parallel})$ and $(L', L'_{\parallel})$ be two framed links such that $lk(L, L_{\parallel}) = lk(L', L'_{\parallel})$. They are respectively framed cobordant to framed knots $(K, K_{\parallel})$ and $(K', K'_{\parallel})$ such that $lk(K, K_{\parallel}) = lk(L, L_{\parallel})$ and $lk(K', K'_{\parallel}) = lk(L', L'_{\parallel})$, so that $lk(K, K_{\parallel}) = lk(K', K'_{\parallel})$. There is a connected cobordism from K to K' that may be equipped with a framing that extends the framing induced by $K_{\parallel}$, and that therefore induces a framing of K' corresponding to a parallel K'_1 of K' such that $lk(K, K_{\parallel}) = lk(K', K'_1)$. Thus $lk(K', K'_1) = lk(K', K'_{\parallel})$ and K'_1 is isotopic to $K'_{\parallel}$, so that $(K', K'_{\parallel})$ is framed cobordant to $(K, K_{\parallel})$. $\diamond$

2.3 More details about the introductions

Let us finish justifying the claims of the introductions.

Lemma 2.13 *For any two transverse sections X and Y of UM, $L_{Y=-X} = -L_{X=-Y}$. For three pairwise transverse sections X, Y and Z of UM, $[L_{Z=-X}] = [L_{Z=-Y}] + [L_{Y=-X}]$ in $H_1(M;\mathbb{Z})$.*

PROOF: For two sections X and Z of UM, transverse to Y, up to homotopy, we can assume that $L_{X=-Y}$ and $L_{Z=-Y}$ are disjoint, and pick disjoint tubular neighborhoods $N(L_{X=-Y})$ and $N(L_{Z=-Y})$ of $L_{X=-Y}$ and $L_{Z=-Y}$, respectively. Then, according to Lemmas 2.4 and 2.5 we can assume that $Z = C(Y, L_{Z=-Y}, \sigma(Z,Y))$ and that $X = C(Y, L_{X=-Y}, \sigma(X,Y))$ so that $Z = Y$ outside $N(L_{Z=-Y})$ and $X = Y$ outside $N(L_{X=-Y})$. Then $L_{Z=-X} = L_{Z=-Y} \coprod L_{Y=-X}$. $\diamond$

Lemma 2.14 *There is a canonical free transitive action of $H_1(M;\mathbb{Z})$ on the set $\mathcal{S}(M)$ of Spin^c-structures of M such that for any two transverse sections Y and Z of UM,*

$$[L_{Z=-Y}][Y]^c = [Z]^c.$$

PROOF: Let Y be a section of UM and let $[K] \in H_1(M;\mathbb{Z})$. Represent $[K]$ by a knot K and equip K with an arbitrary Y-framing. Define $[K][Y]^c$ as $[Z]^c$ with $Z = C(Y, K, \sigma)$. According to Lemma 2.8, $K = L_{Z=-Y}$ and, according to Corollary 2.10, $[Z]^c$ is determined by $[Y]^c$ and $[K]$. Lemma 2.13 ensures that this defines an action of $H_1(M;\mathbb{Z})$. This action is obviously transitive since $[Z]^c = [L_{Z=-Y}][Y]^c$ and Lemma 2.4 ensures that the action is free. $\diamond$

Corollary 2.15 *This action equips $\mathcal{S}(M)$ with an affine structure with translation group $H_1(M;\mathbb{Z})$. With respect to this structure, for two transverse sections X and Y of UM,*

$$[Y]^c - [X]^c = [L_{Y=-X}].$$

Classically, $\mathcal{S}(M)$ is rather equipped with an affine structure with translation group $H^2(M,\partial M;\mathbb{Z})$, and $([Y]^c - [X]^c)_2 \in H^2(M,\partial M;\pi_2(S^2)=\mathbb{Z})$ is the obstruction to homotoping a section Y of UM to another such X over a two-skeleton of M.

Below, we confirm that the two structures are naturally related by the Poincaré duality isomorphism $P\colon H^2(M,\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$.

Lemma 2.16 *For two transverse sections X and Y of UM,*

$$P(([Y]^c - [X]^c)_2) = [Y]^c - [X]^c = [L_{Y=-X}].$$

PROOF: Up to homotopy, assume $Y = C(X, L_{Y=-X}, \sigma(Y,X))$ like in Lemma 2.8. Let S be a 2–chain transverse to $L_{Y=-X}$. We may assume that X and Y coincide outside open disks around $S \cap L_{Y=-X}$. Extend X to a parallelization on the closure of these disks, and see Y as a map from $D^2/\partial D^2$ to S^2 on each of these disks. The sum of the degrees of these maps is the

algebraic intersection of $L_{Y=-X}$ and S. By definition, this is also the evaluation of a cochain that represents $([Y]^c - [X]^c)_2 \in H^2(M, \partial M; \mathbb{Z})$ on S. This shows that $L_{Y=-X}$ is Poincaré dual to $([Y]^c - [X]^c)_2$. $\diamond$

The *Euler class* $e(X^\perp)$ is the obstruction to the existence of a nowhere zero section of $X^\perp$. It lives in $H^2(M;\mathbb{Z})$. In particular, X extends as a parallelization if and only if $e(X^\perp) = 0$. We shall not give a more precise definition for the standard Euler class, since Lemmas 2.17 or 2.18 below can be used as definitions in our cases.

Lemma 2.17 *Let X and Y be two homotopic transverse sections of UM, then $L_{Y=X}$ is Poincaré dual to $e(X^\perp)$. Therefore, $P(e(X^\perp)) = [X]^c - [-X]^c$.*

PROOF: For a section of $X^\perp$, X may be pushed slightly in the direction of the section. If Y denotes the obtained combing, then $L_{Y=X}$ is the vanishing locus of the section that is Poincaré dual to $e(X^\perp)$. $\diamond$

Lemma 2.18 *Let X and Y be two transverse sections of UM,*

$$2[L_{X=Y}] = P(e(X^\perp) + e(Y^\perp))$$

and $2[L_{X=-Y}] = P(e(X^\perp) - e(Y^\perp))$. In particular, for two transverse torsion sections X and Y of UM, $L_{X=Y}$ and $L_{X=-Y}$ represent torsion elements in $H_1(M;\mathbb{Z})$.

PROOF: $[L_{X=Y}] = [X]^c - [-Y]^c = [Y]^c - [-X]^c$ so that $2([L_{X=Y}]) = [X]^c - [-X]^c + [Y]^c - [-Y]^c = P(e(X^\perp) + e(Y^\perp))$. $\diamond$

Lemma 2.19 *Let X and Y be two transverse torsion sections of UM, then $lk(L_{X=Y}, L_{X=-Y})$ only depends on the homotopy classes of X and Y.*

PROOF: Let us prove that $lk(L_{X=Y}, L_{X=-Y})$ does not vary under a generic homotopy of X. Such a homotopy induces two homotopies h_+ and h_- from $[0,1] \times M$ to $S^2 \times S^2$ that are transverse to the diagonal where $h_\pm(t,m) = (X_t(m), \pm Y(m))$. There exists a finite sequence $0 = t_0 < t_1 < t_2 < \ldots < t_k = 1$ of times such that the projections on M of the preimages of the diagonal under $h_{+|[t_i,t_{i+1}]\times M}$ and $h_{-|[t_i,t_{i+1}]\times M}$ are disjoint so that they yield two disjoint cobordisms in M, one from $L_{X_{t_i}=Y}$ to $L_{X_{t_{i+1}}=Y}$, and the other one from $L_{X_{t_i}=-Y}$ to $L_{X_{t_{i+1}}=-Y}$ showing that $lk(L_{X_{t_i}=Y}, L_{X_{t_i}=-Y}) = lk(L_{X_{t_{i+1}}=Y}, L_{X_{t_{i+1}}=-Y})$. $\diamond$

Lemma 2.20 *Let X be a section of UM that extends as a parallelization. The homotopy class of a torsion section Y transverse to X is determined by X, by the homology class $[L_{Y=-X}]$ of $L_{Y=-X}$ in $H_1(M;\mathbb{Z})$, and by the linking number $lk(L_{Y=-X}, L_{Y=X})$.*

PROOF: After a homotopy, Y reads $C(\tau, L_{Y=-X}, L_{Y=X_2})$ where X_2 is the second vector of τ, and, $L_{Y=X}$ and $L_{Y=X_2}$ are parallel knots like in Theorem 2.11. According to Theorem 2.11, the combing $[Y]$ is determined by the framed cobordism class of $L_{Y=-X}$, that is determined by $[L_{Y=-X}]$ and by $lk(L_{Y=-X}, L_{Y=X_2})$ since $L_{Y=-X}$ is rationally null-homologous. After another homotopy that makes Y transverse to X_2 and X, $lk(L_{Y=-X}, L_{Y=X_2}) = lk(L_{Y=-X}, L_{Y=X})$. $\diamond$

2.4 Action of $\pi_3(S^2)$ on combings

Recall that the image of the first basis vector $p_{S^2}\colon SO(3) \to S^2$ induces an isomorphism from $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$ to $\pi_3(S^2)$, where $\tilde{\rho}$ was defined from a map $\rho\colon (B^3, \partial B^3) \to (SO(3), 1)$ before Proposition 2.1. Let γ be the image of $[\tilde{\rho}]$ under this isomorphism. Let X be a combing, extend X to a parallelization (X, Y, Z) on a 3-ball B identified with B^3, and see ρ as a map $\rho\colon (B, \partial B) \to (SO(X, Y, Z), 1)$. Define $\gamma^k X$ as the section that coincides with X outside B and such that

$$\gamma^k X(m \in B) = \rho^k(m)(X)$$

on B. Note that $[\gamma^k X]$ is independent of the chosen parallelization. Since M is connected, any two small enough balls may be put inside a bigger one and $[\gamma^k X]$ is independent of B. Set $\gamma^k[X] = [\gamma^k X]$. It is easy to see that $\gamma^{k+k'}[X] = \gamma^k(\gamma^{k'}[X])$. Let X and Y be two sections of UM that are homotopic except over a 3-ball B^3. Up to homotopy, we may assume that they are identical outside B^3. On B^3, X extends to a parallelization and Y reads as a map from (B^3, S^2) to (S^2, X). It therefore defines an element γ^k of $\pi_3(S^2)$, and $[Y] = \gamma^k[X]$. Thus, $\pi_3(S^2)$ acts transitively on the combings that represent a given Spin^c-structure. In particular it acts transitively on the combings of a $\mathbb{Z}$-sphere.

Lemma 2.21 *Let τ be a parallelization of M and let $[X(\tau)]$ denote the induced combing. Let (U, U_-) be the negative Hopf link*  *($lk(U, U_-) = -1$). Then, with the notation before Theorem 2.11, $[\gamma X(\tau)] = [C(\tau, U, U_-)]$.*

Proof: First note that $[C(\tau, U, U_-)]$ reads $[\gamma^k X(\tau)]$ for an integer k that does not depend on (M, τ). We prove $k = 1$ when $M = B^3$, when τ is the standard parallelization, and when $X = X(\tau)$ is the constant upward vector field, with the help of Lemma 2.20, by showing that

$$lk(L_{\gamma X(\tau) = X'}, L_{\gamma X(\tau) = -X'}) = lk(U, U_-) = -1.$$

for some constant field X' near X. Let N be the North pole of S^2, $(p_{S^2} \circ \rho)^{-1}(N)$ intersects the interior of B^3 as the vertical axis oriented from South to North while $(p_{S^2} \circ \rho)^{-1}(-N)$ intersects B^3 as $\pi \times (-E)$, where E is the equator oriented as a positive meridian of $(p_{S^2} \circ \rho)^{-1}(N)$. Then for N' near N, $lk((p_{S^2} \circ \rho)^{-1}(N'), (p_{S^2} \circ \rho)^{-1}(-N')) = -1$. $\diamond$

Corollary 2.22 *Let τ be a parallelization of M, let $(L, L_{\|})$ be a framed link of L, let (U, U_-) be the negative Hopf link in a ball of M disjoint from L, and let (U, U^+) be the positive Hopf link*  *in a ball of M disjoint from L. Then $[\gamma C(\tau, L, L_{\|})] = [C(\tau, L \cup U, L_{\|} \cup U_-)]$ and $[\gamma^{-1} C(\tau, L, L_{\|})] = [C(\tau, L \cup U, L_{\|} \cup U_+)]$.*

If L is non-empty, let $L_{\|,-1}$ (resp. $L_{\|,+1}$) be a parallel of L obtained from $L_{\|}$ by adding a negative (resp. positive) meridian of L, homologically in $N(L) \setminus L$, then $[C(\tau, L \cup U, L_{\|} \cup U_-)] = [C(\tau, L, L_{\|,-1})]$ and $[C(\tau, L \cup U, L_{\|} \cup U_+)] = [C(\tau, L, L_{\|,+1})]$.

PROOF: Note that $(L, L_{\|,\pm 1})$ is framed cobordant to $(L \cup U, L_{\|} \cup U_{\pm})$ by band sum. Thus, the second formula can be deduced from the fact that the disjoint union of two oppositely framed unknots is framed cobordant to the empty link. $\diamond$

Corollary 2.23 *Let X be a torsion section of UM, let $k \in \mathbb{Z}$ and let Y be a section of UM that represents $[\gamma^k X]$. Then $lk(L_{Y=X}, L_{Y=-X}) = -k$.*

PROOF: We already know that the linking number $lk(L_{Y=X}, L_{Y=-X})$ does not depend on the transverse representatives of $[X]$ and $[Y]$. Furthermore, by Lemma 2.8, $[X]$ can be represented as $C(\tau, L, L_{\|})$ like in Corollary 2.22. Assume $k \neq 0$. Let $(\cup_{i=1}^{|k|} U^{(i)}, \cup_{i=1}^{|k|} U_{\varepsilon}^{(i)})$ denote the union of $|k|$ Hopf links with sign $\varepsilon = -k/|k|$ contained in disjoint balls B_i, for $i = 1, \dots, k$. Let Y be obtained from $C(\tau, L \cup \cup_{i=1}^{|k|} U^{(i)}, L_{\|} \cup \cup_{i=1}^{|k|} U_{-\varepsilon}^{(i)})$ by a small perturbation, induced by the parallelization τ outside $N(L \cup \cup_{i=1}^{|k|} U^{(i)})$ so that it is transverse to X, very close to X, and distinct from $\pm X$ outside $N(L \cup (\cup_{i=1}^{|k|} U^{(i)}))$. Then $L_{Y=-X}$ is a parallel of $\cup_{i=1}^{|k|} U^{(i)}$ and

$$lk(L_{Y=X}, L_{Y=-X}) = \sum_{i=1}^{|k|} lk(L_{Y=X} \cap B_i, L_{Y=-X} \cap B_i) = \sum_{i=1}^{|k|} lk(U^{(i)}, U_{\varepsilon}^{(i)}) = -k.$$

$\diamond$

Proposition 2.24 *Let $[X]^c$ be a Spin^c structure, then the set of combings that belong to $[X]^c$ is an affine space over $\mathbb{Z}/\langle e(X^{\perp}), H_2(M;\mathbb{Z})\rangle$, where the translation by the class of 1 is the action of γ.*

PROOF: Again, fix a parallelization τ of M, and an induced combing Y. This identifies the set $\mathcal{S}(M)$ of Spin^c structures to $H_1(M;\mathbb{Z})$ by mapping $[X]^c$ to the homology class $[L_{X=Y}]$. Any framed link is framed cobordant to a framed knot. According to the Pontrjagin characterization of the combings (Theorem 2.11), the combings that belong to the Spin^c structure $\xi(\tau, [K])$ corresponding to a given class $[K]$ of $H_1(M;\mathbb{Z})$ is the set of framed links homologous to $[K]$ up to framed cobordism. Let K be a knot that represents $[K]$, then all framed links homologous to $[K]$ are framed cobordant to K equipped with some framing, and the combings of $\xi(\tau, [K])$ are the equivalence classes of framings of K up to framed cobordism.

For two parallels K' and K'' of K on the boundary $\partial N(K)$ of a tubular neighborhood $N(K)$ of K, the homology class of $K'' - K'$ in $\partial N(K)$ reads $lk_{N(K)}(K'' - K', K)m(K)$ where $m(K)$ is the oriented meridian of K. The integer $lk_{N(K)}(K'' - K', K)$ measures the difference of the framings induced by K' and K''.

When $[K]$ is a torsion element of $H_1(M;\mathbb{Z})$, the self-linking number $lk(K', K)$ makes sense, and it is a complete invariant of framings of K, up to framed cobordism. This shows that the action of $\pi_3(S^2)$ on the set of combings in a torsion Spin^c-structure is free, and that this set is an affine space over $\mathbb{Z}$.

In general, let B be a cobordism from $0 \times K'$ to $1 \times K''$ in $[0,1] \times N(K)$. Then $lk_{N(K)}(K'' - K', K) = \langle [0,1] \times K, B \rangle_{[0,1]\times M}$. Let C be a framed cobordism from $0 \times K$ to $1 \times K$ in $[0,1] \times M$, and let C' be obtained from C by pushing C in the direction of the framing. Assume that $\partial C' = 1 \times K'' - 0 \times K'$ so that C is a framed cobordism from (K, K') to (K, K'') and

$$0 = \langle C, C' \rangle_{[0,1]\times M} = \langle [0,1] \times K + (C - [0,1] \times K), B + (C' - B) \rangle_{[0,1]\times M}.$$

Since $(C - [0,1] \times K)$ and $(C' - B)$ are 2-cycles in $[0,1] \times M$, $\langle (C - [0,1] \times K), (C' - B) \rangle_{[0,1]\times M} = 0$, and since they are homologous $\langle [0,1] \times K, (C' - B) \rangle_{[0,1]\times M} = \langle (C - [0,1] \times K), B \rangle_{[0,1]\times M}$, so that

$$lk_{N(K)}(K'' - K', K) = -2\langle [0,1] \times K, (C' - B) \rangle_{[0,1]\times M}.$$

In particular, the framing difference induced by C only depends on the homology class of the projection S of C in M, and it is $-2\langle K, S \rangle_M$. Thus if the framings induced by K' and K'' are framed cobordant, $lk_{N(K)}(K'' - K', K)$ is in $\langle 2K, H_2(M;\mathbb{Z}) \rangle_M$. Conversely, for any class S of $H_2(M;\mathbb{Z})$, there exists an embedded connected cobordism C that projects on S. Any framing on $0 \times K$ can be extended to C, and it induces a framing on $1 \times K$, such that the framing difference is $-2\langle K, S \rangle_M$. Since the Euler class of $\xi(\tau, [K])$ is Poincaré dual to $2[K]$, the conclusion follows. $\diamond$

3 The extension of the map p_1

3.1 The key proposition

In this subsection, we prove the following proposition that is the key to the extension of the map p_1.

Proposition 3.1 *Let X, Y and Z be three pairwise transverse torsion sections of UM,*

$$lk(L_{X=Y}, L_{X=-Y}) + lk(L_{Y=Z}, L_{Y=-Z}) = lk(L_{X=Z}, L_{X=-Z}).$$

Consider the 6-manifold $[0,1] \times UM$. Recall that UM is homeomorphic to $M \times S^2$. Let $(S_i)_{i=1,\dots,\beta_1(M)}$ be $\beta_1(M)$ surfaces in the interior of M that represent a basis of $H_2(M;\mathbb{Q})$. For a section Z of UM, let $Z(S_i)$ denote the restriction of Z to S_i. Let $[S]$ denote the homology class of the fiber of UM in $H_2(UM;\mathbb{Q})$, oriented as the boundary of a unit ball of T_xM.

$$H_2(UM;\mathbb{Q}) = \mathbb{Q}[S] \oplus \bigoplus_{i=1}^{\beta_1(M)} \mathbb{Q}[Z(S_i)].$$

Lemma 3.2 *If Y and Z are two transverse sections of UM, then*

$$[Z(S_i)] - [Y(S_i)] = \langle [Z]^c - [Y]^c, S_i \rangle [S] = \langle L_{Z=-Y}, S_i \rangle_M [S]$$

in $H_2(UM;\mathbb{Q})$ (and in $H_2([0,1] \times UM;\mathbb{Q})$).

PROOF: Fix a trivialization of UM so that both Y and Z become functions from M to S^2, then $[Z(S_i)] - [Y(S_i)] = (\deg(Z_{|S_i}) - \deg(Y_{|S_i}))[S]$. If X is a section of UM induced by the trivialization, then $\deg(Z_{|S_i}) = \langle L_{Z=-X}, S_i\rangle_M = \langle [Z]^c - [X]^c, S_i\rangle_M$. $\diamond$

In particular, according to Lemma 2.18, the subspace H_T of $H_2([0,1]\times UM;\mathbb{Q})$ generated by the $[Z(S_i)]$ for torsion combings Z is canonical. Set $H(M) = H_2([0,1]\times UM;\mathbb{Q})/H_T$. Then $H(M) = \mathbb{Q}[S]$.

Let X and Y be two sections of UM. Let $\partial(X,Y)$ be the following codimension 2 submanifold of $\partial([0,1]\times UM)$. If $\partial M = \emptyset$, $\partial(X,Y) = \{1\}\times Y(M) - \{0\}\times X(M)$. If $\partial M = S^2$, let $V(X)$ and $V(Y)$ be the elements of S^2 such that $X = V(X)$ and $Y = V(Y)$ on ∂M. Recall that τ_s identifies $UM_{|S^2}$ with $S^2\times S^2$. Let $P = P(X,Y)$ be a 1–chain in $[0,1]\times S^2$ such that $\partial P = \{1\}\times V(Y) - \{0\}\times V(X)$. Then $\partial(X,Y) = \partial(X,Y,P) = \{1\}\times Y(M) - \{0\}\times X(M) - P\times\partial M$.

Lemma 3.3 *For two transverse sections X and Y of UM such that $([Y]^c - [X]^c)$ vanishes in $H_1(M;\mathbb{Q})$, $\partial(X,Y)$ is rationally null-homologous in $[0,1]\times UM$. It bounds a rational chain $F(X,Y)$ transverse to $\partial([0,1]\times UM)$ that is well-determined, up to the addition of a chain $\Sigma\times\partial M$ for a 2–chain Σ of $[0,1]\times S^2$, up to the addition of a combination of $\{t_i\}\times UM_{|S_i}$ for distinct t_i, and up to cobordism.*

PROOF: $H_3([0,1]\times UM;\mathbb{Q}) \cong H_1(M;\mathbb{Q})\otimes H_2(S^2;\mathbb{Q})$ when $\partial M = S^2$. The direct factor $\mathbb{Q}[X(M)]$ should be added when $\partial M = \emptyset$. The class of a 3–submanifold of $[0,1]\times UM$ vanishes in $H_3([0,1]\times UM;\mathbb{Q})$ if and only if its algebraic intersection with the $[0,1]\times Z(S_i)$ vanishes, for all i, when $\partial M = S^2$, for some combing Z. For $\partial(X,Y)$, this algebraic intersection reads

$$\begin{aligned}\langle [0,1]\times Z(S_i), \partial(X,Y)\rangle_M &= \langle S_i, L_{Z=Y} - L_{Z=X}\rangle = \langle S_i, [Z]^c - [-Y]^c - ([Z]^c - [-X]^c)\rangle \\ &= \langle S_i, [Y]^c - [X]^c\rangle = 0.\end{aligned}$$

When $\partial M = \emptyset$, the algebraic intersection with $[0,1]\times UM_{|\{x\}}$ must vanish, too. This is easily verified. Thus, $\partial(X,Y)$ bounds a rational chain $F(X,Y)$, and since $H_4(UM;\mathbb{Q}) = \bigoplus_{i=1}^{\beta_1(M)}\mathbb{Q}[UM_{|S_i}]$, the second assertion follows. $\diamond$

Lemma 3.4 *For any two transverse torsion sections X and Y of UM, for any two-cycle C of $[0,1]\times UM$, the class of C in $H(M)$ is $\langle C, F(X,Y)\rangle_{[0,1]\times UM}[S]$ for a $F(X,Y)$ as in Lemma 3.3.*

PROOF: First note that $\langle C, F(X,Y)\rangle_{[0,1]\times UM}[S]$ only depends on the homology class of C, for a given $F(X,Y)$, and that $\langle [S], F(X,Y)\rangle = 1$. Now, it suffices to prove that $\langle [Z(S_i)], F(X,Y)\rangle = 0$ for any torsion combing Z, and for any i. Since $\langle [Z(S_i)], F(X,Y)\rangle = \langle [Z(S_i)], X(M)\rangle_{UM} = \langle [Z(S_i)], Y(M)\rangle_{UM}$, $\langle [Z(S_i)], F(X,Y)\rangle$ does not depend on the torsion combings X and Y. In particular, $\langle [Z(S_i)], F(X,Y)\rangle = \langle [Z(S_i)], F(-Z,-Z)\rangle = 0$. $\diamond$

Proposition 3.5 *Let X and Y be two transverse torsion sections of UM. For any $F(X,Y)$ and $F(-X,-Y)$ as in Lemma 3.3, such that the 1–chains $P(X,Y)$ and $P(-X,-Y)$ are disjoint, the class of $F(X,Y)\cap F(-X,-Y)$ in $H(M)$ is*

$$lk(L_{X=Y}, L_{X=-Y})[S].$$

PROOF: Let us first prove that the class of $F(X,Y)\cap F(-X,-Y)$ is well-determined in $H(M)$. When $F(X,Y)$ is changed to $F(X,Y)+(\Sigma\times\partial M)$ for a two-chain Σ of $[0,1]\times S^2$ transverse to $P(-X,-Y)$, $(\Sigma\times\partial M)\cap F(-X,-Y)$ is added to $F(X,Y)\cap F(-X,-Y)$. Now, $(\Sigma\times\partial M)\cap F(-X,-Y)$ is a union of $(t_j,X_j)\times\partial M$ that bounds since the parallelization τ_s extends to M. Thus, the class of $F(X,Y)\cap F(-X,-Y)$ in $H(M)$ in unchanged. Since the class of $\{t_i\}\times UM_{|S_i}\cap F(-X,-Y)$ is in H_T, the class of $F(X,Y)\cap F(-X,-Y)$ is well-determined in $H(M)$.

Now, we construct an explicit $F(X,Y)$ by using the homotopy of Lemma 2.5 that we recall. Assume M is Riemannian. When $X(m)\neq -Y(m)$, there is a unique geodesic arc $[X(m),Y(m)]$ with length ($\ell\in[0,\pi[$) from $X(m)$ to $Y(m)$. For $t\in[0,1]$, let $X_t(m)\in[X(m),Y(m)]$ be such that the length of $[X_0(m)=X(m),X_t(m)]$ is $t\ell$. This defines X_t on $(M\setminus L_{X=-Y})$.

Observe that this definition naturally extends to the boundary of the manifold $M(L_{X=-Y})$ obtained from M by blowing up $L_{X=-Y}$: Trivialize the tangent bundle of M on a tubular neighborhood $N(L_{X=-Y})$ of $L_{X=-Y}$ with a trivialization whose first vector (the image of the North Pole N) is $(-Y)$. Then X maps the normal bundle of $L_{X=-Y}$ to a disk of S^2 around N, by an orientation-preserving diffeomorphism on every fiber (near the origin). In particular, X induces a map from the unit normal bundle of $L_{X=-Y}$ to the unit normal bundle of the North Pole in S^2 that preserves the orientation of the fibers. Then for an element x of the unit normal bundle of $L_{X=-Y}$ in M, $X_t(x)$ describes the half great circle from the North Pole to the South Pole that is tangent to the image of x under the above map. In particular, the whole sphere is covered with degree 1 by the image of ($[0,1]\times$ fiber of the normal bundle). Let G_h be the closure of $\left(\cup_{t\in[0,1]}s_{X_t}(M\setminus L_{X=-Y})\right)$.

$$G_h=\cup_{t\in[0,1]}X_t(M(L_{X=-Y})).$$

Define the 3–cycle of UM

$$p(\partial(X,Y))=Y(M)-X(M)-\partial M\times[V(X),V(Y)]$$

where $[V(X),V(Y)]$ is the shortest geodesic path from $V(X)$ to $V(Y)$ in the fiber of UM over ∂M that is identified with S^2 by τ_s. Then

$$\partial G_h-p(\partial(X,Y))=\cup_{t\in[0,1]}X_t(-\partial M(L_{X=-Y}))=UM_{|L_{X=-Y}}$$

because it is oriented like $\cup_{t\in[0,1]}X_t(\partial N(L_{X=-Y}))$. Let $\Sigma_{X=-Y}$ be a two-chain transverse to $L_{X=Y}$ and bounded by $L_{X=-Y}$ in M. Set $G=G_h-\left(UM_{|\Sigma_{X=-Y}}\right)$ so that $\partial G=p(\partial(X,Y))$. Now let ι be the endomorphism of UM over M that maps a unit vector to the opposite one. Set

$$\begin{array}{rllll} & F(X,Y) & =[0,1/3]\times X(M) & +\{1/3\}\times G & +[1/3,1]\times Y(M)\\ \text{and} & F(-X,-Y) & =[0,2/3]\times(-X)(M) & +\{2/3\}\times\iota(G) & +[2/3,1]\times(-Y)(M).\end{array}$$

Then

$$F(X,Y)\cap F(-X,-Y)=[1/3,2/3]\times Y(L_{Y=-X})-\{1/3\}\times(-X)(\Sigma_{X=-Y})+\{2/3\}\times(Y)(\Sigma_{X=-Y}).$$

Using Lemma 3.4 with $F(X,X) = [0,1]\times X(M)$ to evaluate the class of $(F(X,Y)\cap F(-X,-Y))$ in $H(M)$ finishes the proof. $\diamond$

Proof of Proposition 3.1: Compute $lk(L_{X=Z}, L_{X=-Z})$ by computing the class of $F(X,Z)\cap F(-X,-Z)$ in $H(M)$ where $F(X,Z)$ (resp. $F(-X,-Z)$) is constructed by gluing shrinked copies of $F(X,Y)$ (resp. $F(-X,-Y)$) and $F(Y,Z)$ (resp. $F(-Y,-Z)$) so that $[F(X,Z)\cap F(-X,-Z)] = [F(X,Y)\cap F(-X,-Y)] + [F(Y,Z)\cap F(-Y,-Z)]$. $\diamond$

3.2 Proofs of Theorems 1.2 and 1.1

Lemma 3.6 *Let τ be a trivialization of TM. Let $g \in [(M,\partial M),(SO(3),1)]_m$. Recall that $p_{S^2}\colon SO(3)\to S^2$ maps a transformation t of $SO(3)$ to $t(N)$ where N is the first basis vector of $\mathbb{R}^3$. Let X and Y be two combings of UM induced by τ and $[g][\tau] = [\tau\psi(g)]$, respectively. Then*

$$lk(L_{Y=X}, L_{Y=-X}) = lk((p_{S^2}\circ g)^{-1}(N), (p_{S^2}\circ g)^{-1}(-N)) = -\frac{1}{2}deg(g)$$

Proof: The first equality follows from the definition. It implies that $lk(L_{Y=X}, L_{Y=-X}) = lk((p_{S^2}\circ g)^{-1}(N), (p_{S^2}\circ g)^{-1}(-N)) = lk'(g)$ only depends on g. Then Proposition 3.1 implies that lk' is a homomorphism from $[(M,\partial M),(SO(3),1)]$ to $\mathbb{Q}$. According to Proposition 2.1 it suffices to evaluate it on the element ρ viewed as a degree 2 map of $[(B^3,\partial B^3),(SO(3),1)]_m$. According to Corollary 2.23, when $g=\rho$, $lk(L_{Y=X}, L_{Y=-X}) = -1$. $\diamond$

Proof of Theorem 1.2: Theorem 2.2 and Lemma 3.6 show that if X and Y extend to parallelizations $\tau(X)$ and $\tau(Y)$, then

$$p_1(\tau(Y)) - p_1(\tau(X)) = -4lk(L_{Y=X}, L_{Y=-X}).$$

For any torsion combing $[Y]$, define $p_1([Y])$ from a combing $[X]$ that extends to a parallelization by

$$p_1([Y]) = p_1([X]) + 4lk(L_{X=Y}, L_{X=-Y}).$$

Thanks to Proposition 3.1, since this formula is valid for combings that extend to parallelizations, this definition does not depend on the choice of X. Now, Proposition 3.1 implies that the above formula is valid for all pairs of torsion combings.

Since $[-X] = [X]$ for a section X that extends as a trivialization, we deduce that $p_1([-Y]) = p_1([Y])$, for all torsion sections Y, from the above definition.

According to the following Lemma 3.7, Proposition 2.24 ensures the injectivity of the restriction of p_1 to any torsion Spin^c-structure. $\diamond$

Lemma 3.7 *For any torsion combing $[X]$, $p_1(\gamma[X]) - p_1([X]) = 4$.*

Recall Corollary 2.23. $\diamond$

Proof of Theorem 1.1: The first part of Theorem 1.1 follows from Lemma 2.4 and Corollary 2.10. According to Lemma 2.18, two transverse sections X and Y are torsion sections if and only if $L_{Y=X}$ and $L_{Y=-X}$ are rationally null-homologous. In this case, $lk(L_{Y=X}, L_{Y=-X})$ only depends on the combings $[X]$ and $[Y]$ according to Lemma 2.19. Now, if Y and Y' are such that $L_{Y=X}$ and $L_{Y'=X}$ are homologous, there exists an integer k such that $[Y'] = \gamma^k[Y]$ so that $p_1([Y']) = p_1([Y])+4k$, according to Lemma 3.7, and $lk(L_{Y'=X}, L_{Y'=-X}) = lk(L_{Y=X}, L_{Y=-X})-k$, according to Theorem 1.2. $\diamond$

4 More properties of p_1

4.1 More variations of p_1

Lemma 4.1 *Let M be equipped with a torsion combing X. Let L be a rationally null-homologous link in the interior of M and let Z be a section orthogonal to X of UM, such that Z is defined on L and ∂M. Extend Z as a section $\tilde{Z}$ of the D^2-bundle $X^\perp$, so that $\tilde{Z}$ is transverse to the zero section. Let $L(Z \subset X^\perp)$ be the zero locus of $\tilde{Z}$ cooriented by the fiber D^2 of $X^\perp$. Then $L(Z \subset X^\perp)$ is a link of $M \setminus L$ that represents the Poincaré dual of the relative Euler class of $(X^\perp, Z)$, and $L(Z \subset X^\perp)$ is homologous to the Poincaré dual of $e(X^\perp)$.*

$\diamond$

Remark 4.2 Lemma 4.1 can be taken as a definition of the relative Euler class in this case. The obstruction to extending Z across a 2–cycle of $(M, L \cup \partial M)$ is the intersection of the 2–cycle with $L(Z \subset X^\perp)$.

Proposition 4.3 *Under the assumptions of Lemma 4.1 above, let $\eta = \pm 1$, let $L_\parallel$ be a parallel of L and let $N(L)$ be a tubular neighborhood of L where Z is extended as a section of UM orthogonal to X. For the combing $D(X, L, L_\parallel, Z, \eta)$ of Definition 2.6,*

$$p_1(D(X, L, L_\parallel, Z, \eta)) - p_1(X) = 4lk(L, \eta L(Z \subset X^\perp) - L_\parallel).$$

Proof: Set $Y = D(X, L, L_\parallel, Z, \eta)$. Let τ be the parallelization of $N(L)$ with first vector X and second vector Z. Then τ^{-1} maps $Y(D^2/\partial D^2 \times k)$ to the sphere S^2 with degree $(-\eta)$ so that $L_{Y=-X} = -\eta L$ and $L_{X=-Y} = \eta L$. In order to use Theorem 1.2, deform X to $\tilde{X}$ to make it transverse to Y using $\tilde{Z}$ as follows. Let $N_{1/3}(L) = \{(u\exp(i\theta), k \in L) \in N(L); u \in [0, 1/3]\}$ and $N_{2/3}(L) = \{(u\exp(i\theta), k) \in N(L); u \in [0, 2/3]\}$. Consider a function $\chi\colon M \to [0,1]$ that maps $\left(M \setminus N_{2/3}(L)\right)$ to 1 and $N_{1/3}(L)$ to 0. Let ε be a very small positive real number, set $\tilde{X} = \frac{1}{\|X+\varepsilon\chi\tilde{Z}\|}(X + \varepsilon\chi\tilde{Z})$ so that $\tilde{X}(M)$ is now transverse to $Y(M)$. Outside $UM_{|N(L)}$, $\tilde{X}(M) \cap Y(M)$ reads $Y(L(Z \subset X^\perp))$, whereas on $UM_{|N(L)}$, $Y(M) \cap \tilde{X}(M)$ reads $Y(-\eta L_\parallel)$ because Y covers S^2 with degree $(-\eta)$ along a fiber of $N(L)$. $\diamond$

We have the two immediate corollaries.

Corollary 4.4 *Under the hypotheses of Proposition 4.3, when Z extends as a section of the unit bundle of $X^{\perp}$ on M,*

$$p_1(D(X,L,L_{\parallel},Z,\eta)) = p_1(X) - 4lk(L,L_{\parallel}).$$

Corollary 4.5 *Under the hypotheses of Proposition 4.3, let $K = \{K(\exp(i\kappa) \in S^1)\}$ be a component of L, let $r \in \mathbb{Z}$, and let $Z_r = Z$ on $L \setminus K$ and $Z_r(K(k = \exp(i\kappa))) = \rho(r\kappa, X)(Z)(k)$. Then*

$$p_1(D(X,L,L_{\parallel},Z_r,\eta)) - p_1(D(X,L,L_{\parallel},Z,\eta)) = 4\eta r.$$

Note that under the hypotheses of Proposition 4.3, when X is tangent to L, if Z is induced by $L_{\parallel}$, then $D(X,L,L_{\parallel},Z,1)$ is independent of Z and $L_{\parallel}$.

Proof of Theorem 1.3: Corollary 4.4 shows that the set $p_1(\{\text{Torsion combings}\})$ contains $(p_1(X) - 4\ell(\text{Torsion}(H_1(M;\mathbb{Z}))))$. Conversely, under the assumptions of Theorem 1.3, $p_1(X) - p_1(Y) = 4lk(L_{Y=X}, L_{Y=-X}) \in 4\ell([Y]^c - [X]^c)$ since $[-X]^c = [X]^c$. $\diamond$

The following combing modification also arises in the study of combings associated with Heegaard diagrams.

Proposition 4.6 *Let M be equipped with a torsion combing X. Let $N_0(L)$ denote a tubular neighborhood of a rationally null-homologous link L in the interior of M. Let $L_2 \subset \partial N_0(L)$ be a satellite of L such that the restriction to L_2 of the bundle projection of $N_0(L)$ onto L defines a 2-fold covering of L. Let s be the involution of L_2 that exchanges two points in a fiber of this covering. Pick a parallelization τ of M such that X is constant with respect to τ over $N(L)$. Let Z be a section orthogonal to X of the restriction of UM to L_2, such that $Z(s(k)) = -Z(k)$. Define $D(X,L,L_2,Z,-1)$ as follows. On intervals I of L, trivialize a larger tubular neighborhood $N(L)$ ($N_0(L) \subset N(L)$) as $D^2 \times I$ so that $(D^2 \times I) \cap L_2$ reads $\{-1/2, 1/2\} \times I$, and define $D(X,L,L_2,Z,-1)$ as in Proposition 4.3 on these portions:*

- *$D(X,L,L_2,Z,-1)(0,k \in I) = -X(0,k)$,*
- *when $u \in]0,1]$, $[-X, D(X,L,L_2,Z)(u\exp(i\theta),k)]$ is the geodesic arc of length $u\pi$ of the half great circle $[-X,X]_{\rho(-\theta,X)(Z(1/2,k))}$ from $(-X)$ to X through $\rho(-\theta,X)(Z(1/2,k))$,*

so that $D(X,L,L_2,Z,-1)(1/2,k) = Z(1/2,k)$. Let f be a smooth increasing surjective function from an interval I to $[0,\pi]$, such that all derivatives of f vanish at the ends of I. Let $k \in \mathbb{Z}$. Define

$$\begin{array}{llll} T^k\colon & D^2 \times I & \longrightarrow & D^2 \times I \\ & (u\exp(i\theta),t) & \mapsto & (u\exp(i(\theta + kf(t))),t) \end{array}$$

so that T is a half-twist. Assume that $D^2 \times I$ is a part of $N(L)$ as above and let $(T^k(L_2), T^k_*(Z))$ coincide with (L_2,Z) outside $D^2 \times I$ and read $(T^k(L_2), T^k_*(Z))$ on $D^2 \times I$ where, for $\theta \in \{-1/2, 1/2\}$,

$$T^k_*(Z)((\exp(i(\theta + kf(t)))/2,t)) = \rho(kf(t),X)(Z((\exp(i\theta)/2,t))).$$

Then

$$p_1(D(X,L,T^k(L_2),T_*^k(Z),-1)) - p_1(D(X,L,L_2,Z,-1)) = -4k$$

PROOF: The variation of a combing under some T^k sits inside the ball $D^2 \times I$. Therefore the corresponding variation of p_1 may be read in this ball. It does not depend on the trivialization of the ball induced by X and Z, since all of them are homotopic. Therefore, it only depends on k, linearly. The coefficient is obtained by looking at the effect of the twist T^2 on a $D(X,L,L_{\|},Z,-1)$ like in Proposition 4.3. $\diamond$

4.2 Identifying p_1 with the Gompf invariant

Let us first recall the definition of the Gompf invariant. An *almost-complex structure* on a smooth 4-dimensional manifold W is an operator J such that $J^2 = -1$, acting smoothly on the tangent space of W, fiberwise. An almost-complex structure on W induces a combing of ∂W, that is the class of the image $[JN = J(N(\partial W))]$ under J of the outward normal $N(\partial W)$ of W. Gompf showed that all the combings of a 3-manifold appear as combings JN for some W [Gom98, Lemma 4.4], this will be reproved below. The *first Chern class* $c_1(TW,J)$ of (TW,J) is the obstruction to trivializing TW over the two-skeleton of W as an almost-complex manifold (the induced trivialization of TW must read (X,JX,Y,JY)). The class $c_1(TW,J)$ lives in $H^2(W;\mathbb{Z})$. (The first Chern class c_1 of a complex vector bundle is the Euler class of the corresponding determinant bundle. The reader can check that the definitions coincide in this case.) Its restriction to $H^2(\partial W;\mathbb{Z})$ is $e(JN^\perp)$ so that the boundary of the Poincaré dual $Pc_1(TW,J)$ of $c_1(TW,J)$ is Poincaré dual to $e(JN^\perp)$. When JN is a torsion combing, this boundary $\partial Pc_1(TW,J)$ is a torsion element of $H_1(\partial W;\mathbb{Z})$ so that there exists a rational 2–chain Σ of ∂W such that $(Pc_1(TW,J) \cup \Sigma)$ is a closed rational 2-cycle of W. The algebraic self-intersection of this rational cycle is independent of Σ and it is denoted by $(Pc_1(TW,J))^2$, and the *Gompf invariant* $\theta_G(JN)$ that is denoted by $\theta(JN)$ in [Gom98, Section 4] is

$$\theta_G(JN) = (Pc_1(TW,J))^2 - 2\chi(W) - 3\,\mathrm{signature}(W)$$

where χ stands for the Euler characteristic.

In this subsection, we prove that $\theta_G = p_1$.

Lemma 4.7 *When a combing X of M extends as a parallelization, $\theta_G([X]) = p_1([X])$.*

PROOF: For a rank $2k$ complex bundle ω seen as a rank $4k$ real bundle $\omega_{\mathbb{R}}$, $p_1(\omega_{\mathbb{R}}) = c_1^2(\omega) - 2c_2(\omega)$, where c_2 denotes the second Chern class that is the Euler class of $\omega_{\mathbb{R}}$ for a rank 2 complex bundle ω. See [MS74, Definition p.158 § 14 and Corollary 15.5]. Let (W,J) be an almost-complex connected compact manifold bounded by M such that $X = JN$, let Y be a nowhere zero section of $X^\perp \subset TM$. Consider the complex parallelization (N,Y) inducing the real parallelization (N,JN,Y,JY) of $TW_{|M}$, and the complex bundle ω over $(W \cup_M (-W))$ that is trivial with fiber $\mathbb{C}N \oplus \mathbb{C}Y$ over $(-W)$ and that coincides with the initial one over

W. Since the characteristic classes p_1, c_1 and c_2 of $\omega_{\mathbb{R}}$ or ω trivially restrict to $H^*(-W)$, they come from classes of $H^*(W \cup_M (-W), -W) \cong H^*(W, M)$. Thus, $p_1(\omega)$ is the image of $p_1(W, (JN, Y, JY)) \in H^4(W, \partial W)$, and $c_2(\omega)$ is the image of $c_2(TW, N) \in H^4(W, \partial W)$ that is $\chi(W)[W, \partial W]$ since c_2 is the obstruction to extending N as a nowhere zero section of TW, that is the relative Euler class of (TW, N). Similarly, $c_1(\omega)$ is the image of a lift $\tilde{c}_1$ of $c_1(TW, J)$ in $H^2(W, \partial W)$, where $P\tilde{c}_1$ is represented by a cycle of W. The Poincaré dual $Pc_1(\omega)$ of $c_1(\omega)$ is the image of this cycle in $H_2(W \cup_M (-W))$ and $p_1(W, (JN, Y, JY)) = (Pc_1(TW, J))^2 - 2\chi(W)$. $\diamond$

Lemma 4.8 *When a combing X of M extends as a parallelization, $\theta_G([\gamma X]) = \theta_G([X]) + 4$.*

Proof: According to Lemma 3.7, $p_1([\gamma X]) = p_1([X]) + 4$ for any $[X]$. $\diamond$

Any closed oriented connected 3-manifold M is the boundary of a 4-manifold

$$W_L = B^4 \bigcup_{L \times D^2 \subset S^3} \coprod_{i=1,\dots,n} (D^2 \times D^2)^{(i)}$$

obtained from B^4 by attaching 2-handles $(D^2 \times D^2)^{(i)}_{i=1,\dots,n}$ along a tubular neighborhood $L \times D^2$ of a framed link $L = (K_i, \mu_i)_{i=1,\dots,n}$. Such a framed link L is an *integral surgery presentation* of W_L and M. The K_i are the components of L, the μ_i are the surgery parallels $K_i \times \{1\} \subset K_i \times D^2$ that frame the K_i, and the handle $(D^2 \times D^2)^{(i)}$ is attached by a natural identification of $K_i \times D^2 \subset \partial B^4$ with $((-S^1) \times D^2)^{(i)}$ that restricts to μ_i as an orientation-reversing homeomorphism onto $(S^1 \times \{1\})^{(i)}$.

According to Kaplan [Kap79], we can furthermore demand that $lk(K_i, \mu_i)$ is even for any i, in the statement above. In this case, we shall say that the surgery presentation is *even.*

Lemma 4.9 *Let L be an even surgery presentation of M. There is an almost-complex structure J^0 on W_L (described below) such that $e(J^0 N^\perp) = 0$. For any Spin^c structure ξ on M, there is at least one almost complex structure J on W_L (described below) such that the class of JN belongs to ξ and, if JN is a torsion combing, then $p_1(JN) - p_1(J^0 N) = \theta_G(JN) - \theta_G(J^0 N)$.*

Proof: We shall only consider almost-complex structures J that are *compatible with a* given *Riemannian metric* in the following sense: J preserves the Riemannian metric and Jx is orthogonal to x for any x. Our almost-complex structures J of 4-manifolds also induce the orientation via local parallelizations of the form (X, JX, Y, JY). Below, B^4 is seen as the unit complex ball of $\mathbb{C}^2$, it is equipped with its usual Riemannian structure. It is also seen as the unit ball of the quaternion field $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$.

A homotopy

$$\begin{array}{llll} JN\colon & [-1,0] \times S^3 & \to & TS^3 \\ & (t,x) & \mapsto & JN(t,x) \in T_x S^3 \end{array}$$

such that $JN(-1,x) = ix$, and $\|JN(t,x)\| = 1$ induces a homotopic almost-complex structure on B^4 as follows, the complex structure is unchanged outside a collar $[-1,0] \times S^3$ of the

boundary of B^4, and the operator J of the almost-complex structure maps the unit tangent vector of $[0,1]\times\{x\in S^3\}$ at (t,x) to $JN(t,x)$. Note that J is completely determined by these conditions. If such a homotopy is such that $JN(0,.)$ is tangent to $K_i\times\{y\}$ on $K_i\times D^2$, then the associated almost-complex structure J preserves the tangent space of $\{x\}\times D^2$ and it uniquely extends to $(D^2\times D^2)^{(i)}$ so that J preserves the tangent space of $\{x\}\times D^2$ and J is compatible with the product Riemannian structure on $(D^2\times D^2)^{(i)}$. In particular J maps the outward normal to $(D^2\times S^1)^{(i)}\subset M$ at $(x,y\in S^1)$ to the unit tangent vector of $(\{x\}\times S^1)^{(i)}$ at (x,y).

Before smoothing the ridges, W_L reads $(\mathbb{R}^2\setminus\{(x,y);x<-1,y>-1\})\times(-K_i)\times S^1$ near $K_i\times S^1$. The 4-manifold W_L is next smoothed around $K_i\times S^1$, the smoothing adds the product

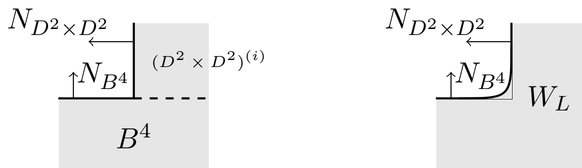

Figure 1: W_L near $K_i\times S^1$ before and after smoothing.

of $K_i\times S^1$ by a triangle with two orthogonal straight sides and a smooth hypothenuse that makes null angles with the two straight sides. See Figure 1.

This new piece may be seen as a part of a $D^2\times\mathbb{R}^2$ that contains $D^2\times D^2$, so that J naturally extends there.

In the plane of the triangle, the normal N reads $N=\cos(\theta)N_{B^4}+\sin(\theta)N_{D^2\times D^2}$ for some $\theta\in[0,\pi/2]$, so that JN reads $JN=\cos(\theta)JN_{B^4}+\sin(\theta)JN_{D^2\times D^2}$ and JN goes from the tangent to $K_i\times\{y\}$ to the tangent of $(\{x\}\times S^1)^{(i)}$ on $T_{(x,y)}K_i\times S^1$ by the shortest possible way on the smooth hypothenuse.

Then J and JN are completely determined on W_L by the homotopy JN on $[-1,0]\times S^3$, and we now study them as a function of this homotopy.

We shall consider homotopies induced by homotopies of orthonormal parallelizations, i.e. homotopies JN such that there is a homotopy $V\colon[-1,0]\times S^3\to T_xS^3$ where $V(t,x)\in T_xS^3$, $V(t,x)\perp JN(t,x)$, $\|V(t,x)\|=1$ and $V(-1,x)=jx$. Furthermore, our homotopies are such that $JN(0,.)$ is tangent to $K_i\times\{y\}$ on $K_i\times D^2$, so that $V(0,x)$ induces a framing of K_i. The linking number of K_i with the parallel of K_i induced by this framing is denoted by r_i. Recall that $H_1(SO(3))=\pi_1(SO(3))=\mathbb{Z}/2\mathbb{Z}$ is generated by a loop of rotation $(\exp(i\theta)\mapsto\rho(\theta,A))$ with a fixed arbitrary axis A.

Let us prove that the integers r_i are odd. Let Σ be a Seifert surface of K_i, then $TM_{|\Sigma}$ has a trivialization τ_Σ whose third vector is the positive normal $N\Sigma$ to Σ, and whose first vector over K_i is obtained from the tangent vector v_K to K_i by rotating it $(-\chi(\Sigma))$ times around the axis $N\Sigma$, along K_i. On the other hand, the first vector of the restriction to K_i of the trivialization τ_{JV} induced by $JN(0,.)$ and $V(0,.)$ is v_K and its third vector is obtained from $N\Sigma$ by rotating it r_i times around v_K along K_i. Then $\tau_\Sigma^{-1}\circ\tau_{JV}$ induces a map from Σ to $SO(3)$

whose restriction to K_i represents a trivial homology class in $H_1(SO(3))$. Since the class of this restriction is $(r_i+\chi(\Sigma))$ mod 2 and since $\chi(\Sigma)$ is odd, r_i is odd, too.

Furthermore, we show later in this proof that any r_i may be changed to any arbitrary odd number, by perturbing the homotopy near $K_i \times D^2$.

Now, the obstruction to extending V as a unit vector tangent to the second almost-complex factor D^2 across $(D^2\times .)^{(i)}$ is $-(r_i - lk(\mu_i, K_i))$, and the obstruction to extending JN that is the tangent to $K_i \times \{y\}$ as a unit vector tangent to the first almost-complex factor across $(D^2\times .)^{(i)}$ is 1. In particular, the Poincaré dual of the Chern class $c_1(TW_L, J)$ may be represented by a chain that does not intersect B^4 and that intersects $(D^2\times D^2)^{(i)}$ as $(1-r_i+lk(\mu_i,K_i))(0\times D^2)^{(i)}$. Let J^0N be a homotopy such that $r_i = lk(\mu_i,K_i)+1$. Then $c_1(TW_L, J^0)=0$ since $H^2(B^4,S^3)=0$ and $\theta_G(J^0N) = -2\chi(W_L) - 3\,\mathrm{signature}(W_L)$.

Assume without loss that $J^0N(0,.)$ is tangent to $K_i\times\{y\}$ on a bigger tubular neighborhood $K_i \times 2D^2$. Let e_1 denote the first basis vector of $\mathbb{R}^3$. Consider a map

$$\begin{array}{llll} F\colon & [0,1]\times\frac{\mathbb{R}}{2\pi\mathbb{Z}} & \to & SO(3) \\ & (t,\theta) & \mapsto & 1 \quad\quad\quad \text{if } t=1 \text{ or } \theta\in 2\pi\mathbb{Z} \\ & & & \rho(2\theta,e_1) \quad \text{if } t=0. \end{array}$$

Composing this map by the evaluation p_{S^2} at e_1 provides a degree ± 1 map from $(D^2,\partial D^2)$ to (S^2,e_1). Then $(J^0N, V^0, J^0V^0)(0,.)$ may be replaced on $K_i\times 2D^2$, by the homotopic

$$(0,(\exp(i\theta),u\exp(i\eta))) \mapsto F(\max(0,u-1),k_i\theta)\left((J^0N,V^0,J^0V^0)(0,(\exp(i\theta),u\exp(i\eta)))\right)$$

for some integer k_i. Since this changes r_i to r_i+2k_i, this shows that r_i can be changed to any odd number.

The obtained almost-complex structure is denoted by J. Let us compare the induced vector fields and compute $L_{JN=V^0}$ and $L_{JN=-V^0}$. We can assume that $L_{JN=V^0}\subset L\times uS^1$ and $L_{JN=-V^0}\subset L\times u'S^1$ for two distinct elements u and u' of $]1,2[$. Then there exists a well-determined $\varepsilon=\pm1$ such that $L_{JN=V^0}$ is homologous to $\varepsilon\sum_{i=1}^n k_i m_i$ in $L\times uS^1$ where m_i is a meridian of K_i in $L\times uS^1$, and $L_{JN=-V^0}$ is homologous to $\varepsilon\sum_{i=1}^n k_i m_{i\|}$ in $L\times u'S^1$, where $m_{i\|}$ is a meridian of K_i in $L\times u'S^1$. In particular, since the meridians m_i generate $H_1(M;\mathbb{Z})$, for any $Spin^c$-structure ξ, there exists an almost-complex structure J as above such that JN belongs to ξ. The combing JN is torsion if and only if $L_{JN=-V^0}$ represents a torsion element in $H_1(M;\mathbb{Z})$. Assume that JN is torsion from now on. Then

$$p_1(JN)-p_1(J^0N) = p_1(JN)-p_1(V^0) = -4lk(\sum_{i=1}^n k_i m_i, \sum_{i=1}^n k_i m_{i\|}).$$

On the other hand, since the boundary of $Pc_1(TW_L,J)$ is homologous to $2L_{JN=-V^0}$, $Pc_1(TW_L,J)$ is represented by $2\varepsilon\sum_{i=1}^n k_i(0\times D^2)^{(i)}$. In order to compute $(Pc_1(TW_L,J))^2$, consider a parallel copy $Pc_1(TW_L,J)_\| = 2\varepsilon\sum_{i=1}^n k_i(x\times D^2)^{(i)}$, and let $(-\partial Pc_1(TW_L,J))$ and $(-\partial Pc_1(TW_L,J)_\|)$

bound Σ and $\Sigma_{\|}$ in M, respectively, so that

$$\begin{aligned}
\theta_G(JN)-\theta_G(J^0N) &= (Pc_1(TW,J))^2\\
&= \langle 2\varepsilon\sum_{i=1}^n k_i(0\times D^2)^{(i)}\cup\Sigma, 2\varepsilon\sum_{i=1}^n k_i(x\times D^2)^{(i)}\cup\Sigma_{\|}\rangle_{W_L\cup_{\partial W_L=0\times M}[0,1]\times M}\\
&= \langle(-[0,1/2]\times\partial\Sigma)\cup(1/2\times\Sigma),(-[0,2/3]\times\partial\Sigma_{\|})\cup(2/3\times\Sigma_{\|})\rangle_{[0,1]\times M}\\
&= -\langle\Sigma,\partial\Sigma_{\|}\rangle_M\\
&= p_1(JN)-p_1(J^0N).
\end{aligned}$$

$\diamond$

The previous lemma, Lemma 3.7 and the transitivity of the action of $\pi_3(S^2)$ on the combings of a $Spin^c$-structure reduce the proof that $\theta_G=p_1$ to the proof of the following lemma.

Lemma 4.10 $\theta_G([\gamma X])-\theta_G([X])=4$ *for any combing* $[X]$.

PROOF: We refer to the previous proof. Add a trivial knot U framed by $+1$ to a surgery presentation L, such that W_L is equipped with an almost-complex structure J. The structure J is homotopic to a structure $J^{(1)}$ that extends on $W_{L\cup U}$ so that $Pc_1(TW,J^{(1)})$ is $(0\times D^2)^{(0)}$. Then $\theta_G(J^{(1)}N)-\theta_G(JN)=1-2-3=-4$. The structure J is also homotopic to a structure $J^{(3)}$ that extends on $W_{L\cup U}$ so that $Pc_1(TW,J^{(3)})$ is $3(0\times D^2)^{(0)}$, then $\theta_G(J^{(3)}N)-\theta_G(J^{(0)}N)=9-2-3=4$. These two combing modifications sit in a 3-ball of M, so that each of them correspond to the action of an element of $\pi_3(S^2)$ independent of (M,J). According to Lemma 4.8, $[J^{(1)}N]=[\gamma^{-1}JN]$ and $[J^{(3)}N]=[\gamma JN]$. Since the above process allows us to inductively represent all the combings $[\gamma^kJN]$, by adding some disjoint trivial knots framed by $+1$, and to prove that $\theta_G(\gamma^kJN)-\theta_G(\gamma^{k-1}JN)=4$, for all $k\in\mathbb{Z}$, we are done. $\diamond$

Remark 4.11 For a natural integer k and for a surgery presentation L of M in S^3, let $L(k)$ be the surgery presentation of M obtained from L by adding k trivial knots framed by $+1$. On our way, we have proved that for any combing $[X]$, for any even surgery presentation L of M, there exists a natural integer k and an almost complex structure J on $W_{L(k)}$ such that $[X]=[JN]$.

5 Relation with the Θ-invariant

5.1 On configuration spaces

Recall that *blowing up* a submanifold A means replacing it by its unit normal bundle.

In a closed 3-manifold R, we fix a point ∞ and define $C_1(R)$ as the compact 3-manifold obtained from R by blowing up $\{\infty\}$. This space $C_1(R)$ is a compactification of $\check{R}=(R\setminus\{\infty\})$.

The *configuration space* $C_2(R)$ is the compact 6–manifold with boundary and corners obtained from R^2 by blowing up (∞,∞), and the closures of $\{\infty\}\times\check{R}$, $\check{R}\times\{\infty\}$ and the diagonal of $\check{R}^2$, successively.

Then $\partial C_2(R)$ contains the unit normal bundle of the diagonal of $\check{R}^2$. This bundle is canonically isomorphic to $U\check{R}$ via the map

$$[(x,y)] \in \frac{\frac{T_r\check{R}^2}{\text{diag}} \setminus \{0\}}{\mathbb{R}^{+*}} \mapsto [y-x] \in \frac{T_r\check{R} \setminus \{0\}}{\mathbb{R}^{+*}}.$$

Since $((\mathbb{R}^3)^2 \setminus \text{diag})$ is homeomorphic to $\mathbb{R}^3 \times]0,\infty[\times S^2$ via the map

$$(x,y) \mapsto (x, \parallel y-x \parallel, \frac{1}{\parallel y-x \parallel}(y-x)),$$

$((\mathbb{R}^3)^2 \setminus \text{diag})$ is homotopy equivalent to S^2. In general, $C_2(R)$ is homotopy equivalent to $(\check{R}^2 \setminus \text{diag})$. When R is a rational homology sphere, $\check{R}$ is a rational homology $\mathbb{R}^3$ and the rational homology of $(\check{R}^2 \setminus \text{diag})$ is isomorphic to the rational homology of $((\mathbb{R}^3)^2 \setminus \text{diag})$. Thus, $C_2(R)$ has the same rational homology as S^2, and $H_2(C_2(R);\mathbb{Q})$ has a canonical generator $[S]$ that is the homology class of a fiber of $U\check{R} \subset C_2(R)$. For a 2-component link (J,K) of $\check{R}$, the homology class $[J \times K]$ of $J \times K$ in $H_2(C_2(R);\mathbb{Q})$ reads $lk(J,K)[S]$, where $lk(J,K)$ is the linking number of J and K, see [Les12, Proposition 1.6].

5.2 On propagators

When R is a rational homology sphere, a *propagator* of $C_2(R)$ is a 4–cycle F of $(C_2(R), \partial C_2(R))$ that is Poincaré dual to the preferred generator of $H^2(C_2(R);\mathbb{Q})$ that maps $[S]$ to 1. For such a propagator F, for any 2-cycle G of $C_2(R)$,

$$[G] = \langle F, G \rangle_{C_2(R)}[S]$$

in $H_2(C_2(R);\mathbb{Q})$.

Let B and $\frac{1}{2}B$ be two balls in $\mathbb{R}^3$ of respective radii ℓ and $\frac{\ell}{2}$, centered at the origin in $\mathbb{R}^3$. Identify a neighborhood of ∞ in R with $S^3 \setminus (\frac{1}{2}B)$ in $(S^3 = \mathbb{R}^3 \cup \{\infty\})$ so that $\check{R}$ reads $\check{R} = M \cup_{]\ell/2,\ell] \times S^2} (\mathbb{R}^3 \setminus (\frac{1}{2}B))$ for a rational homology ball M whose complement in $\check{R}$ is identified with $\mathbb{R}^3 \setminus B$. There is a canonical regular map

$$p_\infty \colon (\partial C_2(R) \setminus UM) \to S^2$$

that maps the limit in $\partial C_2(R)$ of a sequence of ordered pairs of distinct points of $(\check{R} \setminus M)^2$ to the limit of the direction from the first point to the second one. See [Les04a, Lemma 1.1]. Recall that $\tau_s \colon \mathbb{R}^3 \times \mathbb{R}^3 \to T\mathbb{R}^3$ denotes the standard parallelization of $\mathbb{R}^3$. Also recall that the sections X of UM that we consider are *constant* on ∂M, i.e. they read $\tau_s(\partial M \times \{V(X)\})$ for some fixed $V(X) \in S^2$ on ∂M. Let X be such a section. Then the *propagator boundary* ∂F_X associated with X is the following 3–cycle of $\partial C_2(R)$

$$\partial F_X = p_\infty^{-1}(V(X)) \cup X(M)$$

and a *propagator associated with the section* X is a 4–chain F_X of $C_2(R)$ whose boundary reads ∂F_X. Such an F_X is indeed a propagator because the algebraic intersection in UM of a fiber and the section $X(M)$ is one.

5.3 On the Θ-invariant of a combed $\mathbb{Q}$-sphere

Theorem 5.1 *Let X be a section of UM (that is constant on ∂M) for a rational homology ball M, and let $(-X)$ be the opposite section. Let F_X and F_{-X} be two associated transverse propagators. Then $F_X \cap F_{-X}$ is a two-dimensional cycle whose homology class is independent of the chosen propagators. It reads $\Theta(M,[X])[S]$, where $\Theta(M,[X])$ is a rational valued topological invariant of $(M,[X])$.*

PROOF: Recall that $C_2(R)$ has the same rational homology as S^2. In particular, since $H_3(C_2(R);\mathbb{Q}) = 0$, there exist transverse propagators F_X and F_{-X} with the given boudaries ∂F_X and ∂F_{-X}. Without loss, assume that $F_{\pm X} \cap \partial C_2(R) = \partial F_{\pm X}$. Since ∂F_X and ∂F_{-X} do not intersect, $F_X \cap F_{-X}$ is a 2–cycle. Since $H_4(C_2(R);\mathbb{Q}) = 0$, the homology class of $F_X \cap F_{-X}$ in $H_2(C_2(R);\mathbb{Q})$ does not depend on the choices of F_X and F_{-X} with their given boundaries. Then it is easy to see that $\Theta(M,X) \in \mathbb{Q}$ is a locally constant function of the section X. $\diamond$

When R is an integral homology sphere, a combing X is the first vector of a unique parallelization $\tau(X)$ that coincides with τ_s outside M, up to homotopy. When R is a rational homology sphere, and when X is the first vector of a such a parallelization $\tau(X)$, this parallelization is again unique. In this case, the invariant $\Theta(M,X)$ is the degree 1 part of the Kontsevich invariant of $(M,\tau(X))$ [Kon94, KT99, Les04a] and

$$\Theta(M,X) = 6\lambda(M) + \frac{p_1(\tau(X))}{4}.$$

With our extension of the definition of p_1 to combings, we prove that the above formula also holds for combings.

Theorem 5.2 *Let X and Y be two transverse sections of UM. Then*

$$\Theta(M,Y) - \Theta(M,X) = lk(L_{X=Y}, L_{X=-Y}).$$

In particular,

$$\Theta(M,X) = 6\lambda(M) + \frac{p_1(X)}{4}.$$

PROOF: Let us prove that $\Theta(M,Y) - \Theta(M,X) = lk(L_{X=Y}, L_{X=-Y})$. This can be done as follows. Let $F_{-1}(\pm X, \pm Y)$ be the chain $F(\pm X, \pm Y)$ of Lemma 3.3 translated by -1 and seen in a collar $[-1,0] \times UM$ of UM in $C_2(R)$. Assume that F_X and F_{-X} behave as products $[-1,0] \times \partial F_{\pm X}$ in $[-1,0] \times UM$. Then replacing these parts by $F_{-1}(X,Y)$ and $F_{-1}(-X,-Y)$, respectively, and making the appropriate easy corrections in $C_2(R) \setminus C_2(M)$ transforms F_X and F_{-X} into chains F_Y and F_{-Y} so that $[F_Y \cap F_{-Y}] = [F_X \cap F_{-X}] + [F_{-1}(X,Y) \cap F_{-1}(-X,-Y)]$ where $[F_{-1}(X,Y) \cap F_{-1}(-X,-Y)] = lk(L_{X=Y}, L_{X=-Y})[S]$ according to Proposition 3.5. $\diamond$

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